Higher-Order Effects in Asset-Pricing Models with Long-Run Risks*

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November 30, 2015

Abstract

This paper analyzes both the existence of solutions to long-run risk asset pricing models as well as the practicality of approximating these solutions by the Campbell-Shiller log-linearization. We prove a simple relative existence result that is sufficient to show that the original Bansal-Yaron model has a solution. Log-linearization fares less well: we find that for very persistent processes the approximation errors in model moments can be as large as 50%, and can get such basic facts wrong as the direction of the yield curve. The increasing complexity of state-of-the-art asset-pricing models can lead to complex nonlinear solutions with considerable curvature, which in turn can have sizable economic implications. Therefore, these models require numerical solution methods, such as the projection methods employed in this paper, that can adequately describe the higher-order equilibrium features.

Keywords: Asset pricing, long-run risk, log-linearization, nonlinear dynamics, projection methods.

JEL codes: G11, G12.

*This paper was previously circulated under the title "Higher-Order Dynamics in Asset-Pricing Models with Recursive Preferences". We are indebted to Lars Hansen, Ken Judd, and Martin Lettau for helpful discussions on the subject. We thank seminar audiences at the University of Zurich, the Becker Friedman Institute and at various conferences for comments. Karl Schmedders gratefully acknowledges financial support from the Swiss Finance Institute. Ole Wilms gratefully acknowledges financial support from the Zürcher Universitätsverein. Send correspondence to: Karl Schmedders, Department of Business, University of Zurich, Moussonstrasse 15, CH–8044 Zurich, Switzerland; telephone: +41 (0)44 634 3770. E-mail: karl.schmedders@business.uzh.ch.

1 Introduction

This paper presents an analysis of the existence and the computation of solutions to asset-pricing models that feature long-run risk. We provide a formal existence theorem for the basic Bansal and Yaron (2004) long-run risk model. We then show numerically that the model solutions are potentially very nonlinear, and that for many plausible choices of parameters and exogenous processes the errors introduced by linearization are economically significant. In fact, for very persistent processes the approximation errors in model moments can be as large as 50%, and can get such basic facts wrong as the direction of the yield curve. The increasing complexity of state-of-the-art asset-pricing models leads to complex nonlinear equilibrium functions with considerable curvature which in turn can have sizable economic implications. Therefore, these models require numerical solution methods, such as the projection methods employed in this paper, that can adequately describe the higher-order equilibrium features.

Asset-pricing models have become increasingly complex over the last three decades. The first generation of such models, developed in the 1980s (Grossman and Shiller (1981), Hansen and Singleton (1982), Mehra and Prescott (1985)), proved inadequate in explaining large-scale features of financial markets, such as the high equity premium and the low risk-free rate. As the literature on asset-pricing evolved and matured over time, researchers added more and more complex features to their models with, among others, incomplete markets in form of uninsurable income risks, frictions such as borrowing or collateral constraints, time-varying risk aversion, and heterogeneous expectations. While these additional features had varying degrees of success, recently the new generation of long-run risks models (e.g. Bansal and Yaron (2004) or Hansen, Heaton, and Li (2008)) with their interplay of long-run risks, stochastic volatility, and recursive preferences have had considerably more success in resolving long-standing asset pricing puzzles.

An important part of the appeal of the long-run risk model is that Bansal and Yaron (2004) introduce a simple linearized solution method based on the Campbell and Shiller (1988) present-value relation. Long-run risk models feature both highly nonlinear preference structures as well as complex specifications for the exogenous driving forces of the economy. To handle the complexity, researchers must resort to some sort of numerical approximation procedure to make their models tractable. Bansal and Yaron showed that for their original model the log price-dividend ratio could be well-approximated by a linear function of the underlying shocks. The linearized Campbell-Shiller solution, which adjusts for the impact of risk on the average price-dividend ratio, is a considerable advance over the traditional method of log-linearizing around the deterministic steady state, which is known to provide a poor approximation for Epstein-Zin preferences (e.g. Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012), Juillard (2011) or de Groot (2013)).

But time marches on, and researchers have moved with it. By its very nature, a log-linear approximation will miss higher-order effects. Can we always safely ignore these higher-order effects? To answer this question, we examine higher-order dynamics in five additional recent studies, the newly calibrated version of the Bansal and Yaron (2004) model by Bansal, Kiku, and Yaron (2012a), the extensive calibration study of Schorfheide, Song, and Yaron (2014), the volatility-of-volatility model of Bollerslev, Xu, and Zhou (2015) and the work on real and nominal bonds of Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) and Bansal and Shaliastovich (2013).

We show that the errors introduced by the Campbell-Shiller approximation can be large and economically significant. For example, Bansal, Kiku, and Yaron (2012a) recalibrate the original Bansal and Yaron (2004) model to have more persistent shocks to stochastic volatility. We find that for this calibration the log-linearization introduces approximation errors as large as 22% for key quantities such as the equity premium or the volatility of price-dividend ratio. Schorfheide, Song, and Yaron (2014) perform a Bayesian estimation of the model using the same approximation, and find evidence for a higher persistence for long-run risk. In this case we find approximation errors as large as 50% for some key model moments. In general, highly persistent processes lead to solutions that are highly nonlinear, and thus economically relevant approximation errors. Log-linearization can even introduce errors in qualitative conclusions. For example, under high persistence log-linearization can actually invert the slope of the yield curve in the nominal bond models of Bansal and Shaliastovich (2013) and Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010).

As an alternative solution procedure, we use the projection method to solve the nonlinear fixed-point equation for the wealth portfolio. It is known (Atkinson (1992)) that if the fixed-point equation has a solution, then under weak conditions the projection method will converge to a solution. This leads us to consider the question of the existence of a solution. Marinacci and Montrucchio (2010) and Hansen and Scheinkman (2012) prove general theorems about the existence of solutions, but for the types of models considered by the long-run risk literature, the existence of a solution is quite delicate, and depends on specific values of both the preference and exogenous process parameters. We prove a simple *relative* result – if the model has a solution under CRRA preferences for a particular exogenous process specification, then it has a solution for an investor that prefers early resolution for the same specification. For investors that prefer late resolution, the implication goes the other way. We then adapt the results in de Groot (2015) to show that the CRRA version of the long-run risk model has a solution, from which existence for Bansal and Yaron (2004) follows.

Summarizing, by construction, log-linearizing the model as it is commonly done in the asset pricing literature misses higher-order dynamics by construction. If the driving factors of the economy are of low persistence or the risk aversion of the representative agent is low, these

dynamics will have a negligible influence on equilibrium outcomes. However, the combination of highly persistent processes, together with recursive preferences and a risk aversion significantly larger than one, can introduce strong non-linear dynamics to the model. We show that these errors have a large impact on key financial statistics in many recent asset pricing studies and introduce a bias to the model parameters when it comes to estimation or calibration of the model. Therefore, in the future more sophisticated solution methods should be used, as for example projection methods, that can account for higher-order dynamics.

The paper is organized as follows. Section 2 describes the general model framework that is used throughout the paper. In Section 3 we provide a formal theorem for the existence of solution in the economy and analyze the key factors determining existence. Afterwards we examine the effect of higher-order dynamics in six recent asset pricing studies in Section 4. Section 5 concludes.

2 Model Framework

We consider a standard asset-pricing model with a representative agent and recursive preferences as in Epstein and Zin (1989) and Weil (1990). Indirect utility at time t, V_t, is given recursively as

$$V_t = \left[(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta\left[E_t\left(V_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}. \tag{1}$$

In this parametrization, C_t is consumption, δ is the time discount factor, γ determines the level of relative risk aversion, $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$, where ψ is the elasticity of intertemporal substitution (EIS). γ and ψ are required to satisfy $0 < \gamma, \psi$, and $\psi \neq 1$. For $\theta = 1$ the agent has standard CRRA preferences, while $\theta < 1$ indicates a preference for the early resolution of risk and $\theta > 1$ indicates a preference for late resolution. The general asset pricing equation to price any asset i with ex-dividend price $P_{i,t}$ and dividend $D_{i,t}$ is given by

$$E_t\left[M_{t+1}R_{i,t+1}\right] = 1 \tag{2}$$

where $R_{i,t+1} = \frac{P_{i,t+1}+D_{i,t+1}}{P_{i,t}}$. For recursive preferences, the stochastic discount factor M_{t+1} is given by

$$M_{t+1} = \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(\frac{V_{t+1}}{\left[E_t\left(V_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}}}\right)^{\frac{1}{\psi}-\gamma}. \tag{3}$$

Epstein and Zin (1989) show that the (unobserved) value of the aggregate wealth, W_t, can be expressed in terms of the value function,

$$W_t = \frac{V_t^{1-1/\psi}}{(1-\delta)C_t^{-1/\psi}}. \tag{4}$$

This expression in turn permits expressing M_{t+1} in terms of the gross return to the claim on aggregate consumption $R_{w,t+1}$,

$$M_{t+1} = \delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R_{w,t+1}^{\theta-1} \tag{5}$$

where $R_{w,t+1} = \frac{W_{t+1}}{W_t - C_t}$. As equation (2) has to hold for all assets, it must also hold for the return of the aggregate consumption claim. Thus, $R_{w,t+1}$ is determined by the wealth-Euler equation

$$E_t\left[\delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R_{w,t+1}^{\theta}\right] = 1. \tag{6}$$

Throughout the paper we consider a general setup for the specification of log consumption growth, Δc_{t+1}, that allows for long-run risk, x_t, and separate stochastic volatility processes, $\sigma_{c,t}$ and $\sigma_{x,t}$,

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + x_t + \phi_c \sigma_{c,t} \eta_{c,t+1} \\ x_{t+1} &= \rho x_t + \phi_x \sigma_{x,t} \eta_{x,t+1} \end{aligned} \tag{7}$$

where $\eta_{c,t+1}$ and $\eta_{x,t+1}$ are random shocks. In the remainder of the paper we consider variations of this setup that include different specifications for the stochastic volatility processes as well as additional state processes such as volatility of volatility or inflation.

Before we analyze the model we must answer two fundamental questions. First, does a solution for the model exist? Secondly, if a solution exists, how can we reliably compute it? To the best of our knowledge there are no closed-form solutions for the general model. So the common solution approach used in the finance literature is to log-linearize the model, see Segal, Shaliastovich, and Yaron (2015), Bansal, Kiku, and Yaron (2010), Bansal, Kiku, and Yaron (2012a), Bollerslev, Tauchen, and Zhou (2009), Kaltenbrunner and Lochstoer (2010), Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), Drechsler and Yaron (2011), Bansal and Shaliastovich (2013), Constantinides and Ghosh (2011), Bansal, Kiku, Shaliastovich, and Yaron (2014) or Beeler and Campbell (2012), among others. However, log-linearization misses by construction the influence of higher order dynamics; that is, the approach does not attempt

to approximate nonlinear features of the exact solution. But what if these features matter qualitatively for the existence of solutions and quantitatively for equilibrium outcomes? Does log-linearization still deliver sufficiently accurate approximations of the exact solution?

We address these two critical issues in the next section of this paper. We first develop a formal existence criterion for the general model. Once we have established theoretical conditions, we can examine whether the log-linearized solution is in line with the formal existence results. For this task we need a different solution method that accurately accounts for higher-order dynamics and yields robust solutions. A convenient choice are projection methods that allow us to choose the approximation degree as well as the size of the approximation interval in order to be able to capture higher-order elements. While the projection methods require more computational effort, they are capable of correctly capturing higher-order features of the asset returns. For example Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012, p. 189) find that for a stochastic growth model with Epstein-Zin utility projection methods "provide a terrific level of accuracy with reasonable computational burden." We compare the solutions obtained by log-linearization and projection and check whether the log-linearized solution provides reasonably accurate approximations.[1]

3 Existence and Computation of Solutions

Marinacci and Montrucchio (2010) and Hansen and Scheinkman (2012) consider the existence of solutions for the fixed-point equation for the value function for general process specifications. Applying these results to the Bansal-Yaron model has proven delicate. For example, de Groot (2015) considers the existence of solutions for growth economies with stochastic volatility under CRRA preferences, and finds that existence is a complex nonlinear function of the process.

To prove existence, we sidestep the challenge of proving a general result, but instead provide a simple relative result. We show that if the model has a solution for CRRA preferences ($\theta = 1$), then it has a solution when investors have a preference for early resolution of risk ($\theta < 1$), which includes most models in the literature. Interestingly, if investors have a preference for late resolution of uncertainty ($\theta > 1$), the implication is reversed.

This relative result allows us to leverage the extensive literature proving existence for CRRA utility for growth economies. The initial contribution of Burnside (1998) provides both a closed-form solution and characterization for existence when log-consumption growth follows a simple AR(1) process with Gaussian shocks. This result has been extended in various ways. Bidarkota and McCulloch (2003) and Tsionas (2003) generalize the result by

[1] Appendix B provides detailed descriptions of the two solution methods. Appendix C demonstrates the high accuracy of projection methods.

relaxing the assumption of normal shocks to any stable shock distribution and to shocks with well-defined moment generating functions, respectively. Collard, Féve, and Ghattassi (2006) show how to generalize Burnside (1998) to the case of habit formation. Calin, Chen, Cosimano, and Himonas (2005) derive closed-form solutions for asset pricing models with one state variable as long as the utility function and the price-dividend function are analytic. Chen, Cosimano, and Himonas (2008) use this method to analyze existence of solution in the habit model of Campbell and Cochrane (1999) and show how to generalize the approach to multi-dimensional state spaces. Most directly relevant to our application, de Groot (2015) shows how to generalize the result to processes that feature stochastic volatility. In an online appendix, deGroot also provides a closed-form solution for both long-run risk and stochastic volatility, as in the specification of Bansal and Yaron (2004). (The results can be generalized further to specifications featuring, for example, volatility of volatility or inflation.)

3.1 Existence for Epstein-Zin Utility

For the proof of a formal existence theorem for the general model, we first state a special fixed-point result. Subsequently, we present an existence theorem. Appendix A contains the proofs of all formal statements.

3.1.1 A Fixed-Point Result

Marinacci and Montrucchio (2010) apply Tarski's Fixed-Point Theorem, Tarski (1955), to establish the existence of solutions to general nonlinear stochastic equations which encompass, as special cases, many of those arising in stochastic dynamic programming. Here we use a similar fixed-point argument, such as in the proof of Proposition 1 in Marinacci and Montrucchio (2010, Section 4), in a key step towards proving the existence of solutions to the asset-pricing equation for Epstein-Zin utility.

Let s_t be a real vector-valued Markov process with elements in $\mathcal{S} \subset \mathbb{R}^l$, $l \geq 1$, with conditional probability density $p(s'|s)$. Let $\mathcal{V}$ be the set of all Lebesgue-measurable functions $f : \mathcal{S} \to \mathbb{R}_+$, such that

$$\int f(s')^{\max(1,\theta)} p(s'|s) < \infty.$$

This set is the space of all candidate solutions to the fixed-point problems addressed in the lemmata below. We write $f \geq g$ if $f(s) \geq g(s)$ for almost all s. This introduces a partial order on $\mathcal{V}$. With this partial order, for any given $g^* \in \mathcal{V}$, the interval $[0, g^*] \equiv \{f \in \mathcal{V} : 0 \leq f \leq g^*\}$ is a complete lattice.

Now consider a functional $T : \mathcal{V} \to \mathcal{V}$. The functional T is monotone, or order-preserving, iff $f \leq g$ implies $Tf \leq Tg$ for any pair $f, g \in \mathcal{V}$. Further suppose that T maps $[0, g^*]$ to itself

for some $g^* \in \mathcal{V}$, so $T([0, g^*]) \subseteq [0, g^*]$. Then the Tarski Fixed-Point Theorem, Tarski (1955), implies that T has a fixed point in $[0, g^*]$; in fact, the set of fixed points is also a complete lattice. This theorem implies the following lemma.

Lemma 1. *Let $T, U : \mathcal{V} \to \mathcal{V}$, such that for any pair $f, g \in \mathcal{V}$ with $f \leq g$ it holds that*

$$Tf \leq Tg \leq Ug.$$

Further suppose g^* is a fixed point of U. Then T has a fixed point in $[0, g^*]$.

We also need the following lemma.

Lemma 2. *Let $\beta \in [0, 1)$, $\lambda \in \mathbb{R}$, and $0 \neq \theta \leq 1$. Furthermore, let $C, f \in \mathcal{V}$. Let T be the operator*

$$Tf = (1 - \beta)C^{\lambda} + \beta \left[E(f^{\theta}|s)\right]^{1/\theta}$$

and U be the operator

$$Uf = (1 - \beta)C^{\lambda} + \beta E(f|s).$$

Then T and U preserve $\mathcal{V}$. In addition,

(A) Let $0 \neq \theta \leq 1$. Then for any $f, g \in \mathcal{V}$ with $f \leq g$ implies $Tf \leq Tg \leq Ug$.

(B) Let $\theta > 1$. Then for any $f, g \in \mathcal{V}$ with $f \leq g$ implies $Uf \leq Ug \leq Tg$.

Applying Lemma 1 to the operators in Lemma 2 leads to our final conclusion in this section.

Lemma 3. *For the two operators in Lemma 2(A), if U has a fixed point, so does T. For the two operators in Lemma 2(B), if T has a fixed point, so does U.*

This lemma enables us to obtain an existence result for the model with Epstein-Zin (EZ) preferences.

3.1.2 CRRA vs. EZ

Recall the value function recursion (1) for Epstein-Zin utility,

$$V_t = \left[(1 - \delta)C_t^{\frac{1-\gamma}{\theta}} + \delta \left[E_t \left(V_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}$$

with $0 < \gamma, \psi \neq 1$, and $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. The value $\theta = 1$ yields CRRA utility as a special case. Define $\lambda = 1 - \frac{1}{\psi}$ and $\hat{V} = V^{\lambda}$ to obtain

$$\begin{aligned} V_t^{\lambda} &= (1-\delta)C_t^{\lambda} + \delta\left[E_t\left((V_{t+1}^{\lambda})^{\frac{1-\gamma}{\lambda}}\right)\right]^{1/\theta} \\ \iff \quad \hat{V}_t &= (1-\delta)C_t^{\lambda} + \delta\left[E_t\left(\hat{V}_{t+1}^{\theta}\right)\right]^{1/\theta}. \end{aligned} \tag{8}$$

The following theorem relates solutions for the model with CRRA utility ($\theta = 1$) to solutions for the general model with $\theta \neq 1$.

Theorem 1. *Let $0 < \psi \neq 1$ be given. Suppose consumption C is a positive function of a real vector-valued Markov process.*

(A) If the asset-pricing model characterized by equations (1)–(6) has a solution for CRRA utility, $\gamma = \frac{1}{\psi}$, then it also has a solution for Epstein-Zin utility with $0 \neq \theta < 1$; that is, for $1 \neq \gamma > \frac{1}{\psi}$ if $\psi > 1$ and $1 \neq \gamma < \frac{1}{\psi}$ if $\psi < 1$.

(B) If the asset-pricing model characterized by equations (1)–(6) has a solution for Epstein-Zin utility with $\theta > 1$, that is, for $\gamma < \frac{1}{\psi}$ if $\psi > 1$ and $\gamma > \frac{1}{\psi}$ if $\psi < 1$, then it also has a solution for CRRA utility with $\gamma = \frac{1}{\psi}$.

Theorem 1(A) enables us to use existence results for the CRRA case that can be derived for various state-process specifications (see, among others, Burnside (1998) or de Groot (2015)) to determine regions for the parameters ψ and γ for which a solution also exists for EZ utility. The contrapositive of Theorem 1(B) enables us to use the CRRA non-existence results of this literature to determine regions for the parameters ψ and γ for which no solution exists for EZ utility. Since under Epstein-Zin utility, by equation (4),

$$W_t = \frac{\hat{V}}{(1-\delta)C_t^{-1/\psi}},$$

the bound on $\hat{V}$ translate immediately to a bound on wealth, and the wealth-consumption ratio. We will see in our numerical results that this bound is satisfied by the numerical approximations.

Theorem 1 allows for a general consumption process. Next we consider the special state-process specification of Bansal and Yaron (2004). We consider this special specification since it has received much attention in the finance literature. We first analyze the model with CRRA utility and subsequently analyze the implications for general Epstein-Zin preferences using the statements of Theorem 1.

3.2 Existence in the Long-Run Risk Model: CRRA Utility

Bansal and Yaron (2004) use the state processes as in equation (7) with Gaussian shocks and assume that the stochastic volatility in the economy is captured by a single volatility process ($\sigma_{c,t} = \sigma_{x,t} = \sigma_t$):

$$\sigma_{t+1}^2 = \bar{\sigma}^2(1-\nu) + \nu\sigma_t^2 + \phi_\sigma \omega_{t+1}, \tag{9}$$

with $\eta_{c,t+1}, \eta_{x,t+1}, \omega_{t+1} \sim i.i.d.\ N(0,1)$.[2] The following theorem states a formal condition that ensures a finite wealth-consumption ratio and hence the existence of a solution for the model with CRRA utility ($\theta = 1$). Appendix A outlines a proof which closely follows the arguments in de Groot (2015).

Theorem 2. *There exists a solution to model* (1)–(7) *with* $\theta = 1$ *and a single volatility process as specified in equation* (9) *if and only if*

$$\delta \exp\left(\underbrace{B}_{Constant} + \underbrace{B_c\bar{\sigma}^2}_{Consumption\ Shock} + \underbrace{B_x\phi_x^2}_{LRR\ Shock} + \underbrace{B_\sigma\phi_\sigma^2}_{SV\ Shock} \right) < 1, \tag{10}$$

with the following coefficients $\mathcal{B} = (B, B_c, B_x, B_\sigma)$,

$$\begin{aligned}
B &= \mu_c\left(1 - \frac{1}{\psi}\right), \\
B_c &= 0.5\left(1 - \frac{1}{\psi}\right)^2, \\
B_x &= 0.5\left(\frac{1 - \frac{1}{\psi}}{1-\rho}\right)^2 \bar{\sigma}^2, \\
B_\sigma &= \frac{1}{8}\left(\left(\frac{\left(1-\frac{1}{\psi}\right)^2\phi_x^2}{(1-\rho)(1-\nu)}\right)^2 + 2\left(\frac{\left(1-\frac{1}{\psi}\right)^2\phi_x}{(1-\rho)(1-\nu)}\right)^2 + \left(\frac{\left(1-\frac{1}{\psi}\right)^2}{(1-\nu)}\right)^2\right),
\end{aligned} \tag{11}$$

which only depend on the parameters of the state processes and the intertemporal elasticity of substitution, ψ.

Expression (10) shows that the existence of solutions depends on the size of the subjective discount factor δ and a constant part B. In addition, each shock in the model, $\eta_{c,t+1}, \eta_{x,t+1}$

[2] The assumption of normal shocks is not necessary in general for the derivation of closed-form solutions; it suffices that the moment generating function of the shocks exists. For example, de Groot (2015) also provides solutions for a truncated normal and a gamma distribution. These distributions offer the great advantage that the variance process remains positive. However, most of the research following the seminal work by Bansal and Yaron (2004) adopts the normal assumption, which motivates our focus on the existence of solutions for this model class.

and $\eta_{\sigma,t+1}$, adds a new term to the existence requirement.[3] The presence of each type of shock makes the existence requirement more demanding since the three coefficients B_c, B_x, and B_σ are all positive. In the following, we decompose expression (10) to analyze the influence of the three different factors on the existence of solutions. Rewriting the inequality yields

$$B + B_c\bar{\sigma}^2 + B_x\phi_x^2 + B_\sigma\phi_\sigma^2 < -\ln\delta. \tag{12}$$

For the baseline calibration of Bansal and Yaron (2004) with a value of $\delta = 0.998$, the sum over the four components on the left-hand side must be smaller than 0.002. For a larger discount factor δ, as, for example, in the study of Schorfheide, Song, and Yaron (2014) with a value of 0.9996, the condition becomes more stringent with a right-hand side of only 0.0004.

Observe that $\frac{\partial B_x\phi_x^2}{\partial\rho} > 0$, $\frac{\partial B_x\phi_x^2}{\partial\phi_x} > 0$, $\frac{\partial B_\sigma\phi_\sigma^2}{\partial\nu} > 0$ and $\frac{\partial B_\sigma\phi_\sigma^2}{\partial\phi_\sigma} > 0$.[4] Thus the higher the volatility and persistence of the state processes, the more stringent becomes the condition for existence. Table 1 reports the magnitudes of the four terms on the left-hand side of (12) for two different parameterizations. In particular, we provide values for a conservative calibration for the long-run risk process and the stochastic volatility channel and a calibration that takes the more extreme values found in the literature. (Compare Table 2 in Section 4 for the parameter values from six recent studies in the finance literature.)

For a monthly time interval $\mu_c \approx 0.0015$ and the long-run risk literature argues in favor of $\psi \approx 1.5$. These estimates yield a constant term of $B = 0.0005$. For $\psi > 1$, the constant B increases in ψ making the existence condition more stringent as ψ increases. For example, for a value of $\psi = 2$, B becomes 0.0075. Among others, Campbell (1996), Attanasio and Weber (1995) and Yogo (2004) argue for an elasticity of substitution below one. In that case, the constant B becomes negative and hence relaxes the existence condition.

For the conservative parameter range, Table 1 shows that the sum of the four terms on the left-hand side of condition (12) is always (clearly) below 1e-3. Therefore, as long as $\delta < 0.999$, condition (12) easily holds and the model has a solution. For the high parameters estimates the influence of the consumption shock $B_c\bar{\sigma}^2$ is still very small. The influence of the long-run risk process, $B_x\phi_x^2$, strongly increases and assumes values between 0.00013 and 0.02 depending on the EIS ψ. In light of the condition (12), we observe that adding long-run risk to the model can have strong effects on the existence of solutions. The influence of the stochastic volatility shock $B_\sigma\phi_\sigma^2$ remains rather insignificant for an EIS larger than one, but increases strongly as

[3]Note that in this model specification stochastic volatility influences not only shocks to consumption but also shocks to long-run risk. In a more parsimonious setup, with stochastic volatility only entering the shocks to consumption, where the long-run risk factor is a standard AR(1) process ($\sigma_{x,t} = 1, \forall t$) the coefficients simplify to $B_x = 0.5\left(\frac{1-\frac{1}{\psi}}{1-\rho}\right)^2$ and $B_\sigma = \frac{1}{8}\frac{\left(1-\frac{1}{\psi}\right)^4}{(1-\nu)^2}$ and so there is no interaction between the separate terms.

[4]Note that, since stochastic volatility also affects the long-run risk factor, it holds that $\frac{\partial B_\sigma\phi_\sigma^2}{\partial\rho} > 0$ and $\frac{\partial B_\sigma\phi_\sigma^2}{\partial\phi_x} > 0$, making the conditions for existence more stringent as ρ and ϕ_x increase.

Table 1: Existence in the Long-Run Risk Model of Bansal and Yaron (2004)

Conservative Estimates

	B	$B_c\bar{\sigma}^2$	$B_x\phi_x^2$	$B_\sigma\phi_\sigma^2$	Sum
$\psi = 2$	0.00075	6.5e-6	1.5e-5	1.2e-10	0.00077
$\psi = 1.5$	0.00050	2.9e-6	6.7e-6	2.4e-11	0.00051
$\psi = 0.5$	-0.00150	2.6e-5	6.0e-5	1.9e-9	-0.00141
$\psi = 0.2$	-0.00600	4.1e-4	9.6e-4	4.9e-7	-0.00462

High Estimates

	B	$B_c\bar{\sigma}^2$	$B_x\phi_x^2$	$B_\sigma\phi_\sigma^2$	Sum
$\psi = 2$	0.00075	7.6e-6	0.00030	4.9e-6	0.00106
$\psi = 1.5$	0.00050	3.4e-6	0.00013	9.7e-7	0.00064
$\psi = 0.5$	-0.00150	3.0e-5	0.00120	7.8e-5	-0.00019
$\psi = 0.2$	-0.00600	4.9e-4	0.01923	0.0201	0.03381

The table displays values for the four terms in condition (12) *which determine the existence of solutions in the model of Bansal and Yaron (2004) for two sets of parameter calibrations. The conservative parameters are given by* $\bar{\sigma} = 0.0072, \rho = 0.975, \phi_x = 0.038, \nu = 0.956, \phi_\sigma = 2.3e\text{-}6$. *The high estimates are* $\bar{\sigma} = 0.0078, \rho = 0.993, \phi_x = 0.044, \nu = 0.999, \phi_\sigma = 2.8e\text{-}6$. *For both cases we use* $\mu_c = 0.0015$.

an EIS less than one decreases further. In particular, we observe that for $\psi = 0.2$ the model has no solution for $\delta > e^{-0.03381} \approx 0.9668$.

This completes our discussion of existence in the long-run risk model with CRRA preferences. We now combine the insights from Theorems 1 and 2 to analyze the existence of solutions for the model with Epstein-Zin utility.

3.3 Existence in the Long-Run Risk Model: EZ Utility

While there is much debate[5] in the economics and finance literature whether the elasticity of substitution is larger or smaller than one, there appears to be widespread agreement on parameters that satisfy $\gamma > 1/\psi$. Thus, we now restrict attention to models with such preferences. Recall from Theorem 1(A) that, if the model has a solution for CRRA preferences with $\psi > 1$, it also has a solution for recursive preferences with $\gamma > 1/\psi$. And so, for cases such as $\psi = 1.5$ and $\psi = 2$, which we consider in the following, the model with recursive preference with $\gamma > 1/\psi$ has a solution for any exogenous consumption specification satisfying

[5]Table 2 in Section 4 displays EIS values from six recent studies in the asset pricing literature, namely those of Bansal and Yaron (2004), Bansal, Kiku, and Yaron (2012a), Bollerslev, Xu, and Zhou (2015), Schorfheide, Song, and Yaron (2014) and Bansal and Shaliastovich (2013). These studies estimate the long-run risk model by trying to match (asset pricing) moments and obtain values between 1.5 and 2. On the contrary, Yogo (2004) provides estimates below 0.2 using a linearized Euler equation and matching the interest rate. Attanasio and Weber (1995) reports estimates of 0.67 and smaller depending on the data set.

condition (12) in Theorem 2. On the contrary, the contrapositive of Theorem 1(B) shows that, if there is no solution for CRRA preferences with $\psi < 1$, then the model with $\gamma > 1/\psi$ also cannot have a solution. And so, for cases such as $\psi = 0.2$ and $\psi = 0.5$, which we consider in the following as well, the model with recursive preference with $\gamma > 1/\psi$ does not have a solution for any exogenous consumption specification violating condition (12) in Theorem 2.

In the following we illustrate these implications of the two parts of Theorem 1. We first analyze the effects of long-run risk and stochastic volatility on the existence of solutions separately. Subsequently, we examine the existence properties in the full calibrated long-run risk model of Bansal and Yaron (2004).

3.3.1 Long-Run Risk without Stochastic Volatility

To obtain an impression of the isolated effect of long-run risk on the existence of solution with recursive utility, we "shut off" stochastic volatility by setting $\sigma_{c,t} = \sigma_{x,t} = \bar{\sigma}$. Since there is a large debate in the asset pricing literature about the right calibration of the persistence ρ and volatility ϕ_x of the long-run risk process (see for example Bansal, Kiku, and Yaron (2012a), Beeler and Campbell (2012), Schorfheide, Song, and Yaron (2014) and Bollerslev, Xu, and Zhou (2015)), we provide solutions for a range of parameters. Figure 1 shows convergence properties as well as the mean wealth-consumption ratio obtained by the log-linearization and the projection method for a 10×10 grid of values for ϕ_x and ρ. For the case of CRRA utility there are closed-form solutions for the model (see de Groot (2015)) and Theorem 2 shows the formal existence condition. For the case of recursive utility we compute highly accurate solutions using the projection approach.[6] A (green) circle indicates for CRRA utility that the convergence condition of Theorem 2 is satisfied; for EZ utility the circle indicates that both methods produce a solution. A (black) star indicates for CRRA utility that no solution exists and for EZ utility that the projection method does not converge. The lower (blue) values in the figure show the mean wealth-consumption ratio for the log-linearization and the upper (black) values for the projection approach. The entry "Inf" indicates that a method did not find a solution.[7]

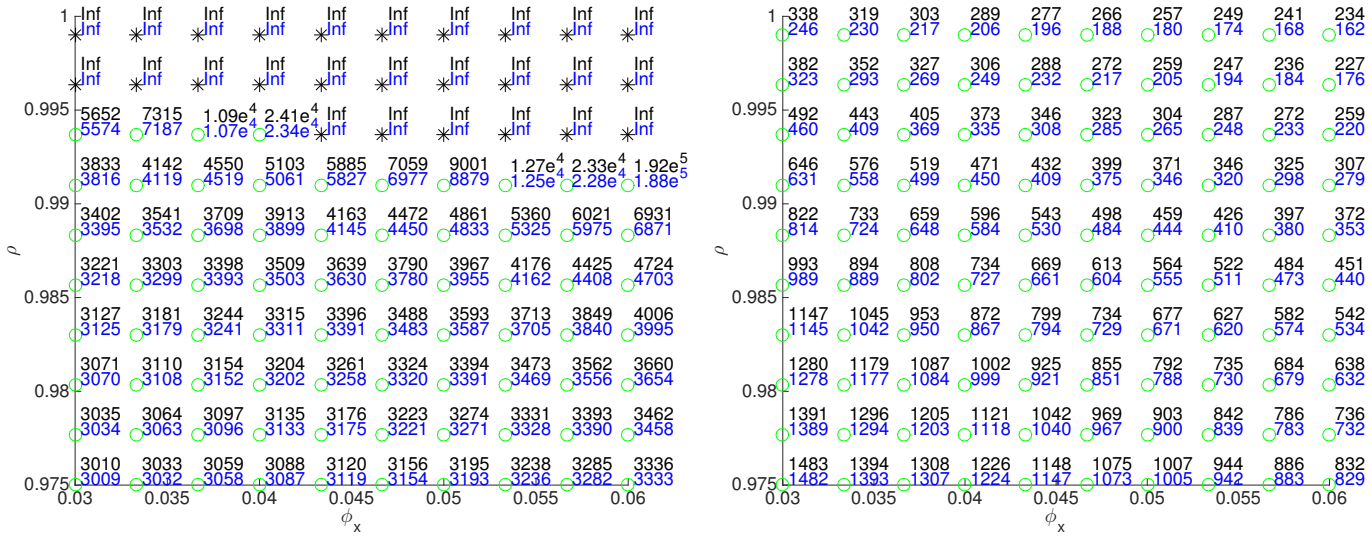
[6] A formal analysis of the accuracy of the projection approach is conducted in Appendix C. To compute accurate solutions with the projection method we increase the approximation interval and the polynomial approximation degree until the solutions no longer change and the polynomial coefficients for the highest degree polynomial are close to zero. By this approach we make sure, that we capture the higher-order dynamics introduced by the tails of the state processes. For the case with CRRA utility we obtain the same solution as the closed-form expressions derived by de Groot (2015) (up to some tiny error). For the cases where there don't exist closed-form solutions we double-checked the accuracy of our computations by using the discretization technique of Tauchen and Hussey (1991) with a very large number of discretization nodes.

[7] Both solution methods ultimately require us to solve a nonlinear system of equations. If the solver cannot solve the system for the log-linearization approach, then, as a robustness check, we attempt to find a solution by setting up a grid of 1000 starting points for the linearization constant. Only if the solver still cannot find a solution, do we report "Inf" for the log-linearization method. If the solver cannot solve the system for the projection method, then we first attempt to compute a solution for a very small state space and a small degree

Figure 1: Influence of Long-Run Risk on Existence and Higher-Order Dynamics

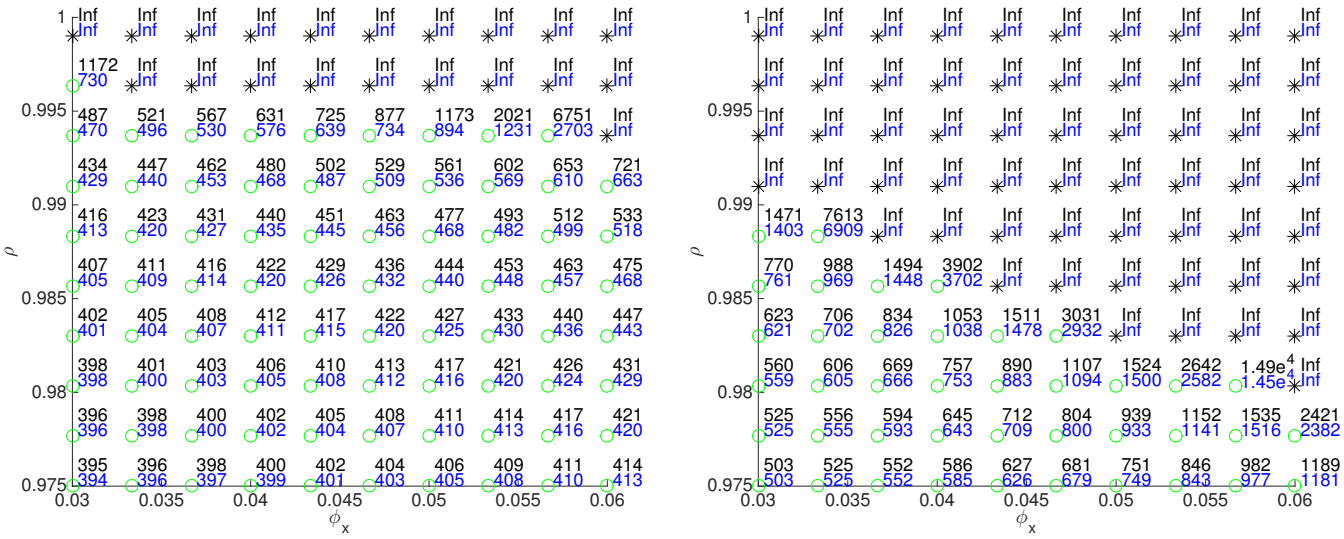
(a) $\psi = 2, \gamma = 0.5$

(b) $\psi = 2, \gamma = 10$



(c) $\psi = 0.5, \gamma = 2$

(d) $\psi = 0.5, \gamma = 10$



The graph shows the convergence properties as well as the mean wealth-consumption ratio for model (7) with constant volatility $\sigma_{c,t} = \sigma_{x,t} = \bar{\sigma}$ and i.i.d. normal shocks $\eta_{c,t+1}$ and $\eta_{x,t+1}$. The results are reported for a range of persistence parameters ρ and volatility parameters ϕ_x. Panels (a) and (c) depict the cases of CRRA utility with $\psi = 2$ and $\psi = 0.5$ respectively, while panels (b) and (d) depict the corresponding cases with EZ utility and $\gamma = 10$. Green circles denote convergence of both, the projection and the log-linearization approach. In the case of CRRA preferences the formal existence condition (10) is also satisfied. Black stars denote cases in which both methods don't converge and the model also doesn't have a solution in the case of CRRA preferences. Black numbers show the mean wealth-consumption ratio obtained by the projection approach and blue numbers show the values obtained by the log-linearization. The remaining model parameters are given by $\delta = 0.9989, \mu = 0.0015, \bar{\sigma} = 0.0078$.

Panel (a) of Figure 1 shows that the model has a solution for CRRA preferences with $\psi = 2$ for sufficiently low values of the volatility parameter and the persistence. In line with Theorem 2, the convergence condition becomes more stringent the larger the persistence, ρ, or the larger the volatility, ϕ_x, and there is no convergence for high-volatility high-persistence combinations. Panel (b) displays the corresponding results for recursive utility with $\psi = 2$, $\gamma = 10$. We find that there is convergence for all parameter combinations on the selected grid. This finding is in line with Theorem 1(A) that, if there exists a solution for the CRRA utility case with $\psi > 1$, there also exists a solution for the model with recursive preferences and $\gamma > 1/\psi$. Put differently, for $\psi > 1$ increasing the risk aversion parameter γ leads to a less stringent existence condition.

Panel (c) of Figure 1 depicts the case of CRRA preferences but with $\psi = 0.5$. Again the model is well behaved in the region of low volatility and low persistence region and the convergence condition becomes more stringent the higher the persistence ρ or the higher the volatility ϕ_x. However, for the corresponding case with recursive preferences and $\gamma = 10$ (Panel (d)), there is no convergence for a much larger set of parameters. Hence, in the case of $\psi < 1$ increasing the coefficient γ makes the existence condition more stringent. This finding is consistent with Theorem 1(B) which shows that the existence condition is more demanding for $\psi < 1$ and $\gamma > 1/\psi$ compared to the respective CRRA case.

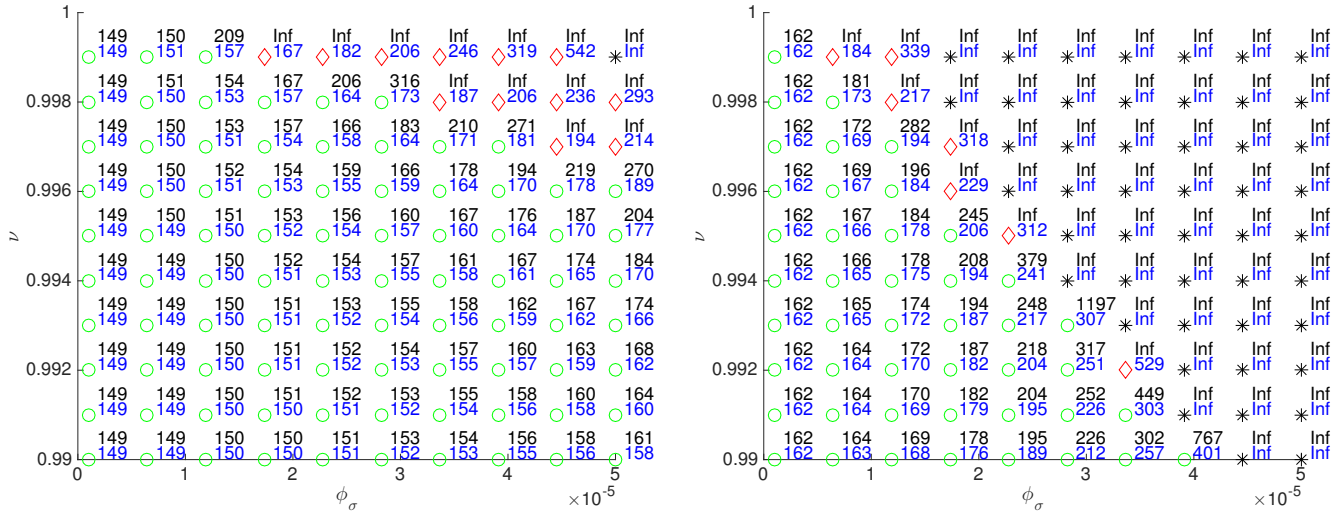
3.3.2 Stochastic Volatility without Long-Run Risk

Figure 2 shows the results for the model with stochastic volatility but without long-run risk ($x_t = 0\ \forall t$). Recall from Table 1 that the influence of the stochastic volatility channel is especially strong for low values of the EIS. Therefore, Panel (a) displays solutions for the CRRA case with an EIS of $\psi = 0.2$ and Panel (b) for the corresponding EZ case with $\gamma = 10$ for a 10×10 grid of values for the parameters ϕ_σ and ν. We now observe a new phenomenon which we represent by (red) diamonds in the figure. For the CRRA case, the diamonds depict parameter combinations for which the model does not have a solution, the condition in Theorem 2 is violated, but the log-linearization approach yields a finite wealth-consumption ratio and incorrectly indicates existence. Simply put, the log-linearization approach delivers a model solution even though the model does not have a(n exact) solution. On the contrary, the projection method correctly indicates nonexistence for all these cases. For the specification with Epstein-Zin utility in panel (b), the diamonds indicate parameter combinations for which the projection method indicates nonexistence while the log-linearization approach indicates

of the approximating polynomial. Subsequently we increase the state space and the polynomial degree. As initial guesses we use solutions from model specifications where we found a solution. While a complete failure of many repeated attempts with the projection method to find a solution are not a proof of non-existence, they give us a high degree of confidence that indeed no solution exists. Also, in the case of CRRA preferences, this approach yields exactly the same convergence results as obtained by the formal Theorem 2.

Figure 2: Influence of Stochastic Volatility on Existence and Higher-Order Dynamics

(a) $\boldsymbol{\psi = 0.2, \gamma = 5}$ (b) $\boldsymbol{\psi = 0.2, \gamma = 10}$



The graph shows the convergence properties as well as the mean wealth-consumption ratio for model (7) with no long-run risk ($x_t = 0\ \forall t$) and a single stochastic volatility process given by equation (9). The results are reported for a range of persistence parameters ν and volatility parameters ϕ_σ. Panel (a) depicts the cases of CRRA utility with $\psi = 0.2$ while panels (b) depicts the corresponding cases with EZ utility and $\gamma = 10$. Green circles denote convergence of both, the projection and the log-linearization approach. In the case of CRRA preferences the formal existence condition (10) is also satisfied. Black stars denote cases in which both methods don't converge and the model also doesn't have a solution in the case of CRRA preferences. Red diamonds denote the cases in which the model doesn't have a solution, but the log-linearization gives a finite wealth-consumption ratio. Black numbers show the mean wealth-consumption ratio obtained by the projection approach and blue numbers show the values obtained by the log-linearization. The remaining model parameters are given by $\delta = 0.9989, \mu = 0.0015, \bar{\sigma} = 0.0072$.

existence.

What is the reason for the failure of the log-linearization approach? Whenever the model does not have a solution, the wealth-consumption ratio is in fact infinite. As we see from the reported values for the mean wealth-consumption ratio in Figure 2, the log-linearization systematically underestimates the wealth-consumption ratio. This underestimation becomes especially strong in the regions close to non-existence leading to fundamentally wrong model outcomes in this parameter region. In addition to this qualitative effect, we also observe a strongly related quantitative effect. For fixed persistence ν of the volatility process, the degree of underestimation increases in the volatility parameter ϕ_σ. That is, the numerical error of the log-linearization result increases in ϕ_σ (until it eventually becomes infinitely large).

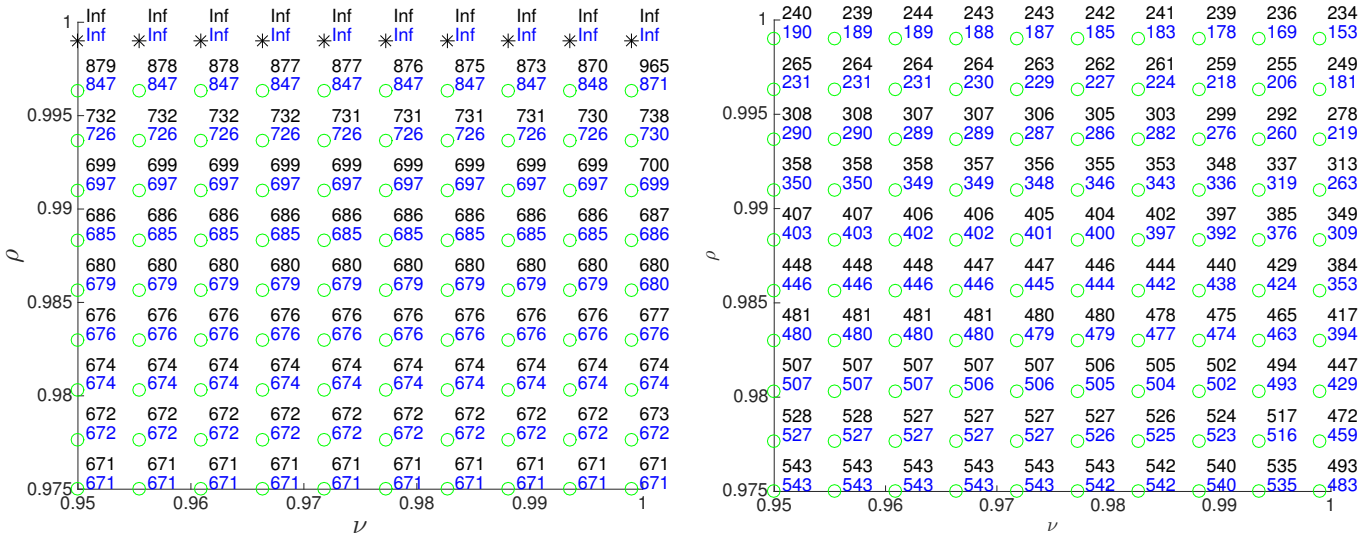
3.3.3 The Long-run Risk Model Calibration of Bansal and Yaron (2004)

In the third and final step of our numerical existence analysis, we show the simultaneous effects of long-run risk and stochastic volatility on the existence of solutions for the long-run risk model of Bansal and Yaron (2004). Figure 3 depicts the convergence properties and the mean wealth-consumption ratios for a grid of values for the persistence parameters of the long run risk process, ρ, and the stochastic volatility, ν. Panel (a) shows results for CRRA utility, while Panel (b) shows results for the utility parameters of Bansal and Yaron (2004). With the exception of models with very high values of the persistence ρ of the long-run risk factor and CRRA utility, the models have a solution. In accordance with Theorem 1(A), increasing the risk aversion to $\gamma = 10$ increases the region of convergence because $\psi > 1$. In line with the results reported in Table 1 in Section 3.2, the stochastic volatility channel does not significantly affect the existence region (the non-existence region does not grow (significantly) with ν) due to its relatively low volatility in the calibration of Bansal and Yaron (2004). Put differently, the additional feature of stochastic volatility in long run risk models has a negligible influence on the qualitative existence issue of solutions. However, the stochastic volatility does have a strong quantitative effect on the approximation errors of the log-linear solution of the model, particularly in Panel (b) which shows the results for the utility parameters of Bansal and Yaron (2004), $\psi = 1.5$, $\gamma = 10$. We observe that both the absolute and the relative difference between the log-linearized and the true model solution increases substantially with both persistence parameters ρ and ν of the long run risk and the stochastic volatility, respectively. Apparently, adding another state process to the model introduces new non-linearities which depend strongly on the persistence of the process.

We emphasize that our analysis of models with very high values for the persistence parameters is not an artificial exercise. In fact, as we report in Table 2 in the next section, recent work on asset pricing models regularly uses highly persistent processes for the exogenous model in-

Figure 3: Existence and Higher-Order Dynamics in the Long-Run Risk Model

(a) $\psi = 1.5, \gamma = 2/3$ (b) $\psi = 1.5, \gamma = 10$



The graph shows the convergence properties as well as the mean wealth-consumption ratio for the long-run risk model of Bansal and Yaron (2004). The results are reported for a range of persistence parameters of the long run risk process ρ and the stochastic volatility process ν. Panels (a) depicts the cases of CRRA utility with $\psi = 1.5$, while panel (b) depicts the corresponding cases with EZ utility and $\gamma = 10$. Green circles denote convergence of both, the projection and the log-linearization approach. In the case of CRRA preferences the formal existence condition (10) is also satisfied. Black stars denote cases in which both methods don't converge and the model also doesn't have a solution in the case of CRRA preferences. Black numbers show the mean wealth-consumption ratio obtained by the projection approach and blue numbers show the values obtained by the log-linearization. The remaining model parameters are given by $\delta = 0.9989, \mu = 0.0015, \bar{\sigma} = 0.0078, \phi_x = 0.044, \phi_\sigma =$ 2.3e-6.

puts. The stochastic volatility and the long-run risk process in Bansal and Yaron (2004), the inflation processes in Bansal and Shaliastovich (2013) and Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) or the different volatility processes in Schorfheide, Song, and Yaron (2014) are a few examples of such processes. In all those papers, log-linearization techniques have been used to analyze equilibrium quantities. But as we have demonstrated above, solving highly persistent models using log-linearization can introduce large approximation errors in the mean wealth-consumption ratio. Naturally now the question arises whether these errors also matter for the model predictions of economically relevant quantities such as, for example, the equity premium, the risk free rate, or return volatilities; or whether perhaps these errors have only small effects on these quantities and so log-linearization remains a reliable solution approach for such model predictions. We answer this question for a number of prominent asset pricing models in the next section.

4 Higher-Order Dynamics in Asset Pricing Models with Recursive Preferences

In this section we compare the implications of the solutions of the log-linearization approach and the projection method for a number of economically relevant quantities. Specifically, we perform this comparison for six different studies from the asset pricing literature on long run risk. The six models are the seminal long-run risk model of Bansal and Yaron (2004), the re-calibrated version of the model by Bansal, Kiku, and Yaron (2012a), the extensive estimation study of Schorfheide, Song, and Yaron (2014), the volatility-of-volatility models of Bollerslev, Tauchen, and Zhou (2009) and Bollerslev, Xu, and Zhou (2015), and the two studies study of real and nominal bonds of Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) and Bansal and Shaliastovich (2013). Common to all these studies is the methodological attempt to match several key statistics on financial markets such as the high equity premium, a low risk-free rate, volatile stock prices, real and nominal bond prices, the volatility premium or patterns in return predictability. Obviously, in order to determine a reasonable calibration of the model it is essential to solve the model without significant errors in the approximation of those key statistics since such errors could potentially bias the calibration.

In the previous section, we have seen that the log-linearization approach produces sizable approximation errors for the mean wealth-consumption ratio in the long-run risk model of Bansal and Yaron (2004). Now we show that these errors carry forward to substantial errors in the first and second moments of asset returns. In fact, we demonstrate that making use of the log-linearization approach has a strong impact on the financial market statistics implied by the models.

4.1 Six Model Specifications

The models share the same basic model setup (7) augmented with a process for log dividend growth Δd_{t+1} that is potentially correlated with consumption,

$$\begin{aligned}\Delta c_{t+1} &= \mu_c + x_t + \phi_c \sigma_{c,t} \eta_{c,t+1} \\ x_{t+1} &= \rho x_t + \phi_x \sigma_{x,t} \eta_{x,t+1} \\ \Delta d_{t+1} &= \mu_d + \Phi x_t + \phi_d \sigma_{d,t} \eta_{d,t+1} + \phi_{d,c} \sigma_{c,t} \eta_{c,t+1} \\ \eta_{c,t+1}, \eta_{x,t+1}, \eta_{d,t+1} &\sim i.i.d.\ N(0,1).\end{aligned} \tag{13}$$

Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012a) assume that there is a single volatility process that drives uncertainty in the economy $\sigma_{c,t} = \sigma_{x,t} = \sigma_{d,t} = \sigma_t$ with

$$\sigma_{t+1}^2 = \bar{\sigma}^2(1-\nu) + \nu\sigma_t^2 + \phi_\sigma \omega_{t+1} \qquad \omega_{t+1} \sim i.i.d.\ N(0,1). \tag{14}$$

Schorfheide, Song, and Yaron (2014) relax this assumption by allowing for three separate volatility processes. The two volatility processes for consumption growth and the long-run risk factor are required to account for the weak correlation between the risk-free rate and consumption growth. As shown in their estimation study, the volatility dynamics of dividends differs significantly from the other two processes. Therefore, a third process is required to model the stochastic volatility of dividends. Schorfheide, Song, and Yaron (2014) assume that the logarithm of the volatility process is normal to ensure that the standard deviation of the shocks remains positive,

$$\begin{aligned}\sigma_{i,t} &= \varphi_i \bar{\sigma} \exp(h_{i,t}) \\ h_{i,t+1} &= \nu_i h_{i,t} + \sigma_{h_i} \sqrt{1-\nu_i^2}\, \omega_{i,t+1}, \qquad i \in \{c, x, d\} \\ \omega_{i,t+1} &\sim i.i.d.\ N(0,1).\end{aligned} \tag{15}$$

In order to derive analytical solutions for the log-linearization coefficients that are needed for their estimation study, Schorfheide, Song, and Yaron (2014) use a linear approximation of the volatility dynamics that follows Gaussian dynamics,

$$\sigma_{i,t}^2 \approx 2(\varphi_i \bar{\sigma})^2 h_{i,t} + (\varphi_i \bar{\sigma})^2 \tag{16}$$

which in turn yields

$$\sigma_{i,t+1}^2 = \bar{\sigma}_i^2(1-\nu_i) + \nu_i \sigma_{i,t}^2 + \phi_{\sigma_i} \omega_{i,t+1}$$

with $\phi_{\sigma_i} = 2\bar{\sigma}_i^2 \sigma_{h_i} \sqrt{1-\nu_i^2}$ and $\bar{\sigma}_i = \varphi_i \bar{\sigma}$.[8]

[8] We proceed in the same way as Schorfheide, Song, and Yaron (2014) by solving the model using the linearized

The fourth model stems from the estimation study of Bollerslev, Xu, and Zhou (2015). In a standard long-run risk model with stochastic volatility many long-standing puzzling behaviors on financial markets such as a high equity risk premium together with a low risk-free rate, volatile price dynamics or predictability of stock returns can be explained. However, the most recent research has gone one step further by showing that the standard model is not able to generate a time-varying variance risk premium that has predictive power for stock returns. Fortunately, the literature has also suggested a possible solution for this puzzle by including time-varying volatility of volatility (vol-of-vol) to the model, see, for example, Bollerslev, Tauchen, and Zhou (2009), Tauchen (2011), Drechsler and Yaron (2011), Bollerslev, Xu, and Zhou (2015) or Dew-Becker, Giglio, Le, and Rodriguez (2015). Bollerslev, Xu, and Zhou (2015) consider a slight variation of the long-run risk factor compared to the baseline model (13) where the vol-of-vol factor q_t drives the volatility,[9]

$$\begin{aligned} \sigma_{t+1}^2 &= \bar{\sigma}^2(1-\nu) + \nu\sigma_t^2 + \phi_\sigma\sqrt{q_t}\omega_{\sigma,t+1} \\ q_{t+1} &= \mu_q(1-\rho_q) + \rho_q q_t + \phi_q\sqrt{q_t}\omega_{q,t+1} \\ x_{t+1} &= \rho x_t + \phi_x\sqrt{q_t}\eta_{x,t+1} \\ &\eta_{x,t+1}, \omega_{\sigma,t+1}, \omega_{q,t+1} \sim i.i.d.\ N(0,1). \end{aligned} \tag{17}$$

The vol-of-vol factor q_t follows a square root process. This process specification has also been used, for example, in Tauchen (2011) or the seminal work on volatility of volatility in this model class by Bollerslev, Tauchen, and Zhou (2009). However, a square root process poses a new challenge to the model, as the process can become complex when q_t becomes negative. This problem is usually circumvented by assuming a reflecting boundary at zero to ensure positivity. (In fact, this approach has also been used for the stochastic volatility process in the original Bansal and Yaron (2004) study and many subsequent papers in the long-run risk literature.) However, for a simple computation of model solutions, the assumption of a non-truncated distribution for the log-linearization is commonly used. In Appendix D we analyze in more detail how the square-root process specification and the issue of complexity affects the log-linearized solution. In particular we find that for the calibration in Bollerslev, Tauchen, and Zhou (2009) equilibrium model solutions are not real numbers but instead are complex numbers. For the parameters in Bollerslev, Xu, and Zhou (2015) the process is centered well above zero and the standard log-linearization technique yields a real solution. Therefore, we

version of the volatility dynamics to obtain quasi-closed form solutions for the linearization coefficients; for the inference of moments we use the original specification to ensure that the volatility of the model stays positive.

[9] Drechsler and Yaron (2011) use a similar model where the volatility of x_t is driven by σ_t instead of q_t, see their 2007 working paper version. However, Bollerslev, Xu, and Zhou (2015) provide evidence for a better empirical match for their model specification. The estimation study of Bollerslev, Xu, and Zhou (2015) also models cross-correlations between the shocks of the state processes. For the analysis of the non-linear dynamics of the model we keep the model as parsimonious as possible and drop the cross-correlations.

concentrate on this calibration in the main text.

The fifth study under consideration is the work on real and nominal bonds and the size of the martingale component in the stochastic discount factor by Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010). They add inflation π_t with a stochastic growth rate $x_{\pi,t}$ to the standard model (13) and price nominal bonds[10]

$$\begin{aligned}
\pi_{t+1} &= \mu_\pi + x_{\pi,t} + \phi_{\pi,c}\sigma_{c,t}\eta_{c,t+1} + \phi_{\pi,x}\sigma_{x,t}\eta_{x,t+1} + \sigma_\pi\eta_{\pi,t+1} \\
x_{\pi,t+1} &= \mu_{x_\pi}(1-\rho_\pi) + \rho_\pi x_{\pi,t} + \rho_{\pi,x}x_t \\
&\quad + \phi_{x_\pi,c}\sigma_{c,t}\eta_{c,t+1} + \phi_{x_\pi,x}\sigma_{x,t}\eta_{x,t+1} + \sigma_{x_\pi}\eta_{\pi,t+1} \\
&\quad \eta_{\pi,t+1} \sim i.i.d.\ N(0,1).
\end{aligned} \tag{18}$$

Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) assume that there are two stochastic volatility processes for consumption growth and the long-run risk component ($\sigma_{d,t} = \sigma_{c,t}$)

$$\sigma^2_{i,t+1} = \bar{\sigma}^2_i(1-\nu_i) + \nu_i\sigma^2_{i,t} + \phi_{\sigma_i}\omega_{i,t+1}, \qquad i \in \{c,d\},$$

and inflation, the stochastic growth rate of inflation and dividends have loadings on these two volatility channels.

The sixth and last study under consideration is the subsequent work on nominal and real bonds of Bansal and Shaliastovich (2013). The setup is very similar to Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) but they assume that $x_{\pi,t}$ enters the real stochastic growth rate of consumption x_t to model the non-neutral effect of expected inflation on future expected growth,

$$\begin{aligned}
\pi_{t+1} &= \mu_\pi + x_{\pi,t} + \sigma_\pi\eta_{\pi,t+1} \\
x_{\pi,t+1} &= \rho_\pi x_{\pi,t} + \sigma^\pi_t e_{\pi,t+1} \\
x_{t+1} &= \rho x_t + \rho_{x\pi}x_{\pi,t} + \sigma^x_t e_{x,t+1} \\
&\quad \eta_{\pi,t+1}, e_{\pi,t+1}, e_{x,t+1} \sim i.i.d.\ N(0,1).
\end{aligned} \tag{19}$$

Also they assume that there is a separate AR(1) process for the volatility of the stochastic growth rate of inflation σ^π_t and the volatility of consumption growth is constant ($\sigma_{c,t} = \bar{\sigma}_c$):

$$\sigma^2_{i,t+1} = \bar{\sigma}^2_i(1-\nu_i) + \nu_i\sigma^2_{i,t} + \phi_{\sigma_i}\omega_{i,t+1}, \qquad i \in \{x,\pi\}.$$

As the focus of Bansal and Shaliastovich (2013) is on bond markets, they do not include a process for dividends.

Table 2 lists the parameter values of the six studies.[11] While the parameters in Bansal

[10] The model setup is the same as in the 2008 version of Bansal and Shaliastovich (2013). In the paper they write $\bar{\pi}_t$ for $x_{\pi,t}$.

[11] For the model of Bollerslev, Xu, and Zhou (2015) we use the parameters estimates in the study for ρ, ν and

Table 2: Model Parameters

		BY (2004)	BKY (2012)	SSY (2014)	BS (2013)	KLVV (2010)	BXZ (2015)
Preferences	γ	10	10	10.84	20.90	8	10
	ψ	1.5	1.5	1.7	1.81	1.5	1.5
	δ	0.998	0.9989	0.9996	0.994	0.9987	0.999
Consumption	μ_c	0.0015	0.0015	0.0016	0.0049	0.0016	0.0015
	ϕ_c	1	1	1	1	1	0.00546
	ρ	0.979	0.975	0.993	0.81	0.991	0.988
	ϕ_x	0.044	0.038	1	1	1	3.12e–4
	$\rho_{x\pi}$	–	–	–	–0.047	0	–
Volatility	ν_c	0.987	0.999	0.956	0	0.85	0.64
	ν_x	–	–	0.99	0.994	0.996	–
	ν_d	–	–	0.94	–	-	–
	ν_π	–	–	–	0.979	-	–
	ϕ_{σ_c}	2.3e–6	2.8e–6	8.8e–6	0	1.15e–6	1
	ϕ_{σ_x}	–	–	6.0e–9	1.85e–7	4.19e–9	–
	ϕ_{σ_d}	–	–	2.3e–4	–	-	–
	ϕ_{σ_π}	–	–	–	1.81e–7	-	–
	$\bar{\sigma}_c$	0.0078	0.0072	0.005	4.6e–3	0.004	1
	$\bar{\sigma}_x$	–	–	2.0e–4	1.09e–3	1.60e–5	–
	$\bar{\sigma}_d$	–	–	0.0273	–	-	–
	$\bar{\sigma}_\pi$	–	–	–	1.11e–3	-	–
Dividends	μ_d	0.0015	0.0015	0.001	–	0.0015	0.0015
	Φ	3.0	2.5	3.2	–	1.5	3.0
	ϕ_d	4.5	5.96	1	–	6	0.0246
	$\phi_{d,c}$	0	2.6	1.17	–	0.6	0
Inflation	μ_π	–	–	–	0.0090	0	–
	μ_{x_π}	–	–	–	0	0.0032	–
	σ_π	–	–	–	0.0055	0.0035	–
	σ_{x_π}	–	–	–	0	4e–6	–
	$\phi_{\pi,c}$	–	–	–	0	0	–
	$\phi_{\pi,x}$	–	–	–	0	-2	–
	$\phi_{x_\pi,c}$	–	–	–	0	0	–
	$\phi_{x_\pi,x}$	–	–	–	0	-1	–
	ρ_π	–	–	–	0.988	0.83	–
	$\rho_{\pi,x}$	–	–	–	0	-0.35	–
Vol–of–Vol	μ_q	–	–	–	–		0.211
	ϕ_q	–	–	–	–		0.632
	ρ_q	–	–	–	–		0.46

Parameter values as reported in the studies of Bansal and Yaron (2004), Bansal, Kiku, and Yaron (2012a), Schorfheide, Song, and Yaron (2014), Bansal and Shaliastovich (2013), Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), and Bollerslev, Xu, and Zhou (2015).

and Yaron (2004) and Bansal, Kiku, and Yaron (2012a) are calibrated, Schorfheide, Song, and Yaron (2014), Bollerslev, Xu, and Zhou (2015) and Bansal and Shaliastovich (2013) estimate the model parameters to match annual financial market characteristics. In the first five models the investor has a monthly decision interval, while Bansal and Shaliastovich (2013) use quarterly intervals. This distinction explains, for example, the considerable difference in the level parameters. The main difference between the sets of parameter of the original Bansal and Yaron (2004) calibration and the new calibration of Bansal, Kiku, and Yaron (2012a) is that in the new calibration, the persistence of the volatility shock, ν_c, is higher and that shocks to dividends are correlated with short-run shocks to consumption growth ($\phi_{d,c} = 2.6$ in the new calibration compared to $\phi_{d,c} = 0$ in the original calibration). These changes increase the influence of the volatility channel compared to the long-run risks channel of the model. The adjustment is needed to get rid of some implications of the original calibration that are inconsistent with the data. In particular, as, for example, Zhou and Zhu (2015) or Beeler and Campbell (2012) point out for the original 2004 calibration, the log price-dividend ratio has predictive power for future consumption growth, while this relationship is not present in the data. By increasing the influence of the volatility channel, this predictability vanishes.

Schorfheide, Song, and Yaron (2014) provide further evidence for a highly persistent stochastic growth rate $\rho = 0.993$ with a 90% confidence interval of $\{0.989, 0.994\}$. In line with the calibrated values in Bansal, Kiku, and Yaron (2012a), they also find a highly persistent volatility process for the long-run risk component, while the estimates for consumption and dividend volatility are slightly smaller.

4.2 Moments and Errors

Table 3 reports annualized summary statistics and numerical errors for the five models that include a dividend process. The reported financial statistics are the mean and standard deviation of the price-dividend ratio, the averages of the market excess return and the risk-free return, and the volatilities of the excess return and the risk-free rate.[12] The table reports these statistics for both the solution of the log-linearization approach and the projection

ρ_q. As they do not report values for the remaining parameters, we use the calibration as reported in the 2007 working paper version of Drechsler and Yaron (2011).

[12] We solve the model for the return of the wealth portfolio, z_w, the market portfolio, z_m, and the risk-free rate, z_{rf}. To compute the annualized moments, we simulate 1,000,000 years of artificial data. Beeler and Campbell (2012) provide a detailed description of how to compute the annual moments from the monthly observations. A significant issue in the model is that the variance process σ_t^2 can, in fact, become negative. To overcome this problem, Bansal and Yaron (2004) replace all negative realizations with very small but positive values. We proceed in the same way for both methods to achieve consistent results. For the approximation interval of the projection methods we choose the interval to be slightly larger than the maximum observation range of the long simulations. As in the previous section, we increase the polynomial degree until the coefficients of the highest-order polynomial are close to zero. We double-check the accuracy of the solution by increasing the approximation interval until the solutions do not change.

approach; in addition, it states the relative errors induced by the linearization.

Table 3: Annualized Moments and Errors

	$E(p_t - d_t)$	$\sigma(p_t - d_t)$	$E\left(r_t^m - r_t^f\right)$	$E\left(r_t^f\right)$	$\sigma(r_t^m)$	$\sigma\left(r_t^f\right)$
Bansal and Yaron (2004)						
Log-Lin	3.1749	0.2012	4.61	1.46	17.05	1.31
Projection	3.2056	0.1990	4.48	1.46	16.97	1.31
Error	0.96 %	1.12%	2.93%	0.08%	0.50%	0.05%
Bansal, Kiku, and Yaron (2012a)						
Log-Lin	3.0473	0.2910	5.73	0.99	21.27	1.28
Projection	3.2413	0.2389	4.69	1.10	21.00	1.27
Error	5.98%	21.81%	22.26%	10.21%	1.28%	1.45%
Schorfheide, Song, and Yaron (2014)						
Log-Lin	1.9394	0.3331	18.00	-1.80	20.43	1.56
Projection	2.3497	0.2892	12.00	-1.17	18.43	1.39
Error	17.46%	15.18 %	50.02%	54.54%	10.84%	12.29%
Bollerslev, Xu, and Zhou (2015)						
Log-Lin	2.7479	0.2485	7.27	1.16	16.28	1.36
Projection	2.8225	0.2399	6.78	1.17	15.91	1.35
Error	2.64%	3.58%	7.26%	0.72%	2.35%	0.40%
Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010)						
Log-Lin	3.1102	0.1782	4.85	1.64	11.53	1.17
Projection	3.3468	0.1465	3.56	1.32	10.58	1.14
Error	7.07%	21.66%	36.29%	19.43%	9.07%	2.52%

The table shows the mean and the standard deviation of the annualized log price-dividend ratio, the annualized market over the risk-free return and the risk-free return. Results obtained by the log-linearization and the projection method as well as the relative error of the log-linearization are shown for the models of Bansal and Yaron (2004), Bansal, Kiku, and Yaron (2012a), Schorfheide, Song, and Yaron (2014), Bollerslev, Xu, and Zhou (2015) and Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010). All returns are shown in percent, so a value of 1.5 is a 1.5% annualized figure.

We observe that the log-linearization does a reasonably good job for the parameters in Bansal and Yaron (2004) with a maximal error of 2.93% for the equity premium. For the parameter set of Bansal, Kiku, and Yaron (2012a) the results are considerably worse. The log-linearization overstates the equity premium by more than 100 basis points; and it predicts a volatility of the log price-dividend ratio of 0.2910 instead of 0.2389. These values correspond to relative errors of about 22%. Simply put, the log-linearization falsely produces a large equity premium and volatile log price-dividend ratio even though the true model solution is significantly smaller.

For the model of Schorfheide, Song, and Yaron (2014) approximation errors become even larger. For the equity premium and the risk-free rate the errors exceed 50% and also the errors in the other four key statistics exceed 10%.[13] In Section 4.4 below, we carve out the source for these large numerical errors. It is the interplay of the highly persistent state processes that introduces substantial non-linearities to the model solutions; as a result, even a slight increase in the persistence parameter of the long-run risk channel can dramatically increase the approximation errors of the log-linearized solution. Schorfheide, Song, and Yaron (2014) estimate a persistence of $\rho = 0.993$ compared to $\rho = 0.975$ in the calibration of Bansal, Kiku, and Yaron (2012a) which explains the large approximations errors. Hence, using the log-linearized solution to estimate models featuring highly persistent state processes can potentially introduce a large bias to the implied model moments and so, in turn, biases the estimation results for the model parameters.

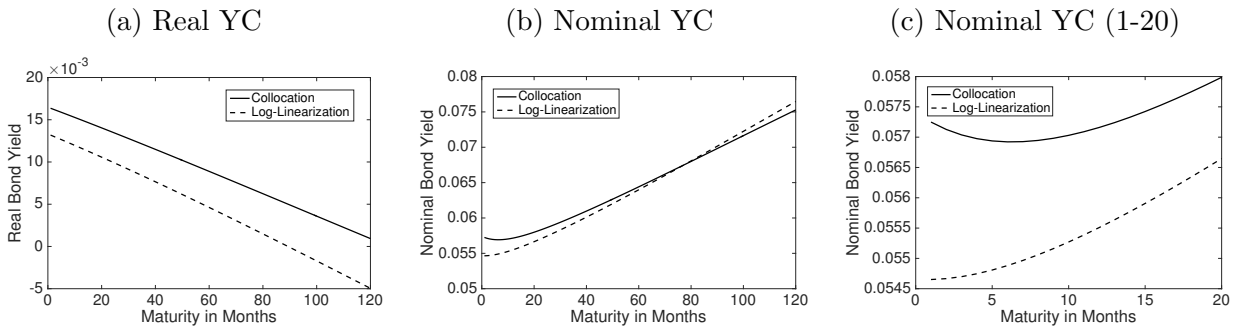
This finding is in line with the results for the model of Bollerslev, Xu, and Zhou (2015). The model only features a highly persistent long-run risk process $\rho = 0.988$ while the persistence parameters of the stochastic volatility and vol-of-vol factors are rather low ($\nu = 0.64$ and $\rho_q = 0.46$). Consequently the approximation errors are rather small with a maximum error of 7.26% for the equity premium. This result is not surprising as the authors mention in their estimation that the stochastic volatility and the vol-of-vol factors only influence the variance premium and have a negligible influence on the price and return dynamics. Concordantly, we obtain almost the same results when setting the volatility of the two factors to zero ($\phi_\sigma = \phi_q = 0$).

For the study of Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) we also find large errors with a maximum error in the equity premium of 36.29%. An overestimation of the premium of more than 100 basis points. Their calibration features a highly persistent long-run risk process $\rho = 0.991$ and highly persistent stochastic volatility of long-run risk $\nu_x = 0.996$ that introduce the large non-linearities to the model. Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) not only analyze equity markets but also price real and nominal bonds to analyze the martingale component in the stochastic discount factor. In Figure 4 we show the real and nominal yield curve for their model.

We find that the differences between the yield curve obtained by linearizing the model and solving it accurately using the projection approach are small in absolute values. However, the nominal yield curve from the linearized model differs in its shape. While the true nominal yield

[13] Note that the results are very sensitive to changes in the model parameters. Here we show model outcomes for the median estimates of Schorfheide, Song, and Yaron (2014), while in the original study they draw parameter values from the estimated distributions of the model parameters and report the median for a large number of draws. For example, for the 5% quantile estimates the model yields an equity premium of 2.4% with a risk-free rate of 2.3%. This explains why the values reported here differ from the values shown in Table 4 of the study of Schorfheide, Song, and Yaron (2014).

Figure 4: Real and Nominal Yield Curve in the Model of Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010)

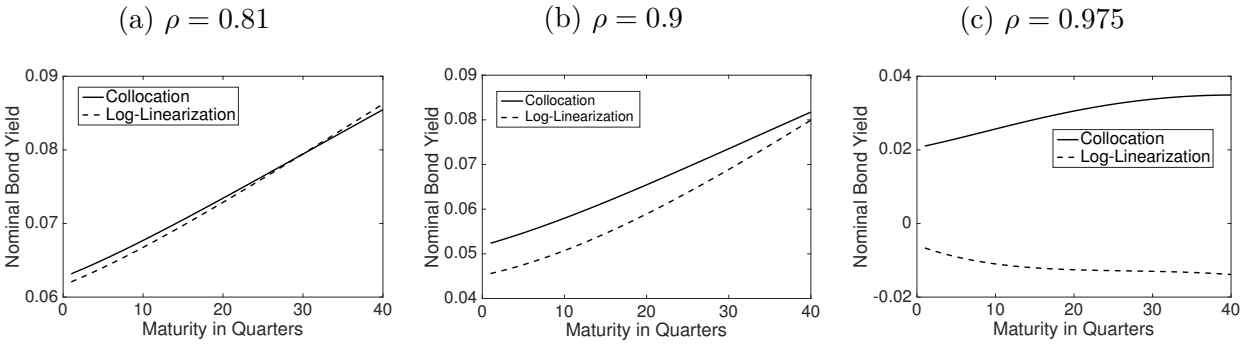


The graph shows the yield curves for real and nominal bonds in the model of Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010). Panel (c) shows the yield curve for 1-20 months bonds.

curve is downwards sloping in the short run and upwards sloping in the long run, this pattern does not occur when using log-linearization. So linearizing the model potentially affects the shape of the real curve.

The work of Bansal and Shaliastovich (2013) provides further insights to this finding. In Figure 5 we show the nominal yield curve in their model.

Figure 5: Nominal Yield Curve in the Model of Bansal and Shaliastovich (2013)



The graph shows the yield curve for nominal bonds in the model of Bansal and Shaliastovich (2013).

Panel (a) shows the yield curve for the parameters in the original study. We observe that the difference between the log-linearized solution and the projection solution is negligible with very small errors and also the shape of the yield curve is correct. As Bansal and Shaliastovich (2013) use bond data to estimate the model, they find a very low persistence in the long-run risk component with $\rho = 0.81$. This comparably low amount of persistence makes it difficult to match key moments for equity markets. For example the annualized equity premium for their parameter estimates is only 1.69%.[14] Therefore we increase ρ in panels (b) and (c) to 0.9

[14]The published version of Bansal and Shaliastovich (2013) does not provide a process for dividend growth. For

and 0.975 correspondingly to increase the premium paid for long-run consumption risk.[15] We find that the errors in the yield curve grow significantly as ρ approaches the value 1. In fact, for $\rho = 0.975$ the log-linearization predicts a downward sloping nominal yield curve (dashed line) even though the model actually produces an upward sloping curve (solid line). Hence, relying on the log-linearization to solve the model can lead to false conclusions not only about the magnitude of bond yields but even about the shape of the yield curve.

In sum, we observe that while the log-linearization approach produces satisfactory solutions for an analysis of the models in Bansal and Yaron (2004) and Bollerslev, Xu, and Zhou (2015), the method performs rather poorly for the models in Bansal, Kiku, and Yaron (2012a), Schorfheide, Song, and Yaron (2014), and Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010). For these latter models, the poor approximations have a strong effect on the model predictions for key financial statistics. Our observations motivate the next step in our analysis. We want to understand which model characteristics affect the performance of the log-linearization approach; simply put, when can we trust the results of such an approach and when can we not? And related to this question, we also want to understand which properties of the exact solution lead to a poor performance of a linear method; that is, what exactly goes wrong with the linearized solution?

4.3 The Interplay of the State Processes

The log-linearization approach assumes that, on the state space of the model, the first derivatives of the solution are approximately constant and the second derivatives are approximately zero. We now show numerically that this assumption fails to hold for models with more than one highly persistent state process. We demonstrate that for solutions of such models the second derivatives can be very large, the interplay of state process leads to highly nonlinear solutions; and so higher-order effects matter for the predictions of such models. The sizable deviations from linearity in the models' solutions is the cause for the failure of the log-linearization approach.

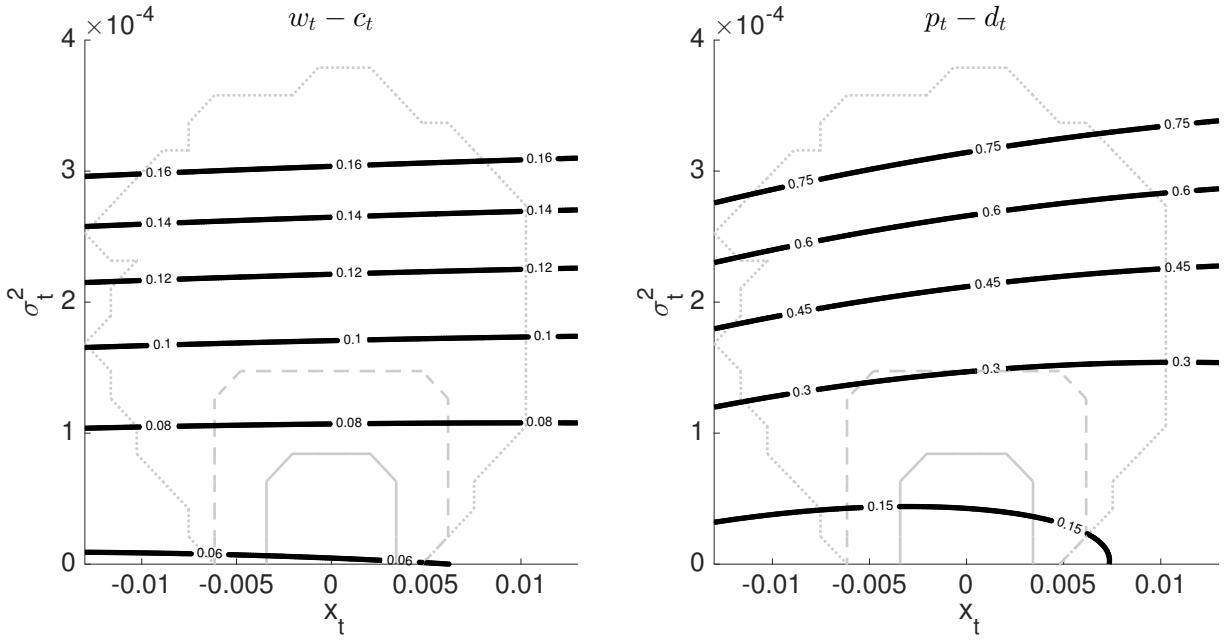
For the purpose of making these points, we concentrate on the two fundamental factors of long-run risk and stochastic volatility. We use the calibration of Bansal, Kiku, and Yaron (2012a) (see equations (13)-(14)). Figure 6 shows isolines for the absolute errors in the log wealth-consumption ratio (left panel) and the log price-dividend ratio (right panel) of the log-linearization as a function of the states x and σ^2 (black solid lines). For example along

the purposes of comparison, we consider the specification that appears in the 2007 working paper of their paper. The process for Δd_{t+1} is the same as in Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) (see equation 18). As the 2007 working paper assumes a monthly decision interval and the published version from 2013 has a quarterly interval, we adjust the volatility of dividends ϕ_d to match the volatility of dividend growth in the data of approximately 11% annualized.

[15] For $\rho = 0.9$ we obtain an equity premium of 4.48% and for $\rho = 0.97$ a premium of 10.57%.

a line marked with '0.1', the absolute error of the log-linearization is 0.1. The figure also shows the regions into which 50%, 90% and 100% of the observations fall. These regions show the subsets of the state space that the model actually visits and in which regions it "spends most of its time" during long simulations. Corresponding errors for the first derivatives with respect to the state variables are shown in Figure 7 and for the second derivative in Figure 8.

Figure 6: Approximation Errors in the log Wealth-Consumption and log Price-Dividend Ratio of the Log-Linearization

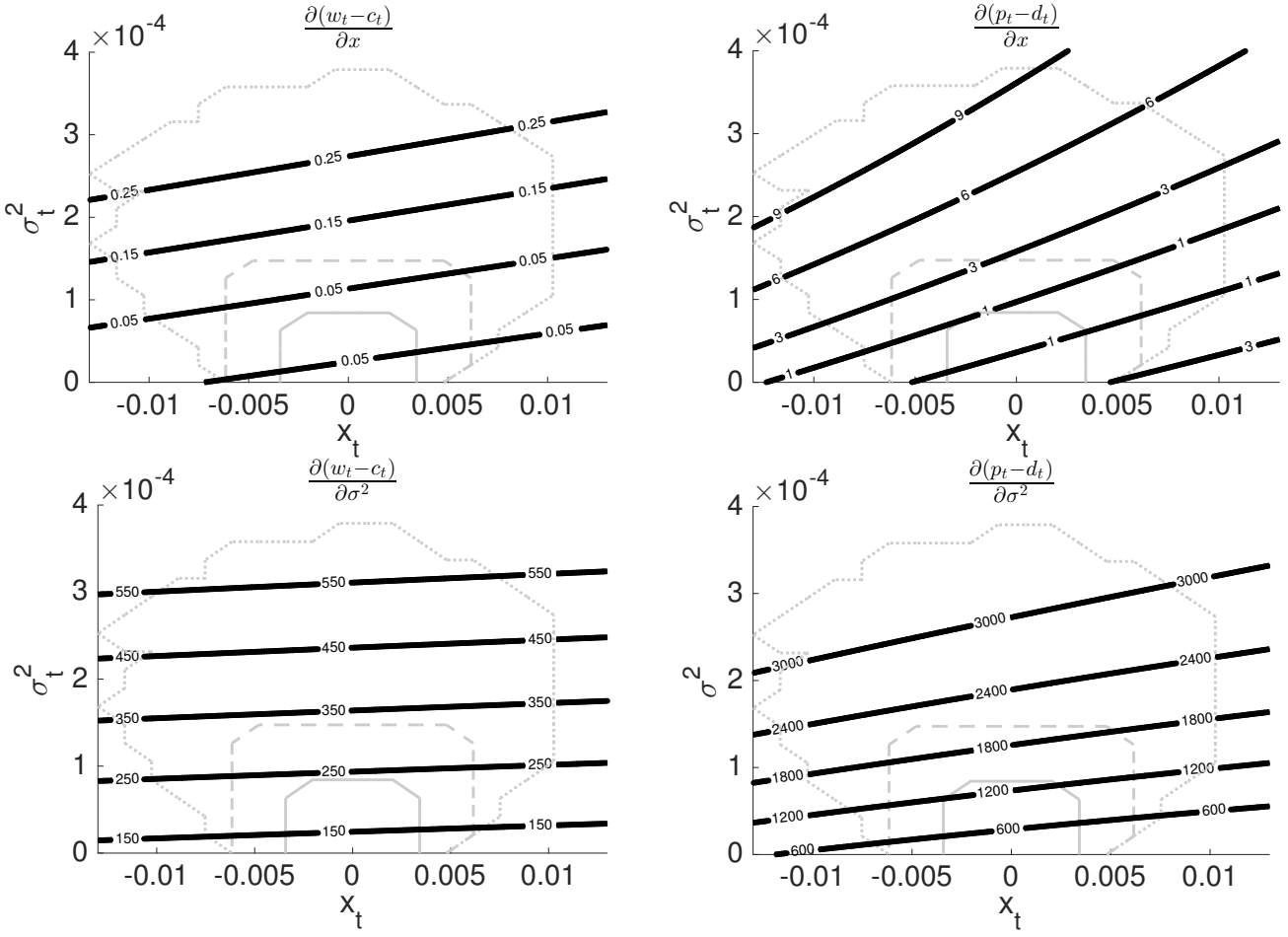


The graph shows isolines for the absolute errors in the log wealth-consumption ratio (left panel) and the log price-dividend ratio (right panel) of the log-linearization as a function of the states x and σ^2 (black solid lines). The (grey) dotted, dashed and solid lines mark the respective areas into which 100%, 90% and 50% of the observations from 10^6 simulated data points fall. The parameter values are from the calibration of Bansal, Kiku, and Yaron (2012a), see Table 2.

We find that the errors in the log wealth-consumption are rather small with maximum values of about 0.16 within the observation range. For the log price-dividend ratio, the errors are also small in the area close to the long-run mean of the processes, but they increase significantly with σ^2 and reach values of up to 0.3 in the 90% observation range, see Figure 6. Put differently, the price dividend ratio obtained by the log-linearization is off by a factor of $e^{0.3} \approx 1.35$ for almost 10% of the time and can be off by a factor larger than 2 for extreme values reached in the simulations.

The errors in the first derivatives show similar patterns. Again the errors in the derivatives of the price-dividend ratio are significantly larger than the errors in the derivatives of the wealth-consumption ratio and the errors increase monotonically with σ^2 for the BKY (2012)

Figure 7: Approximation Errors in the First Derivatives of the log Wealth-Consumption and log Price-Dividend Ratio of the Log-Linearization



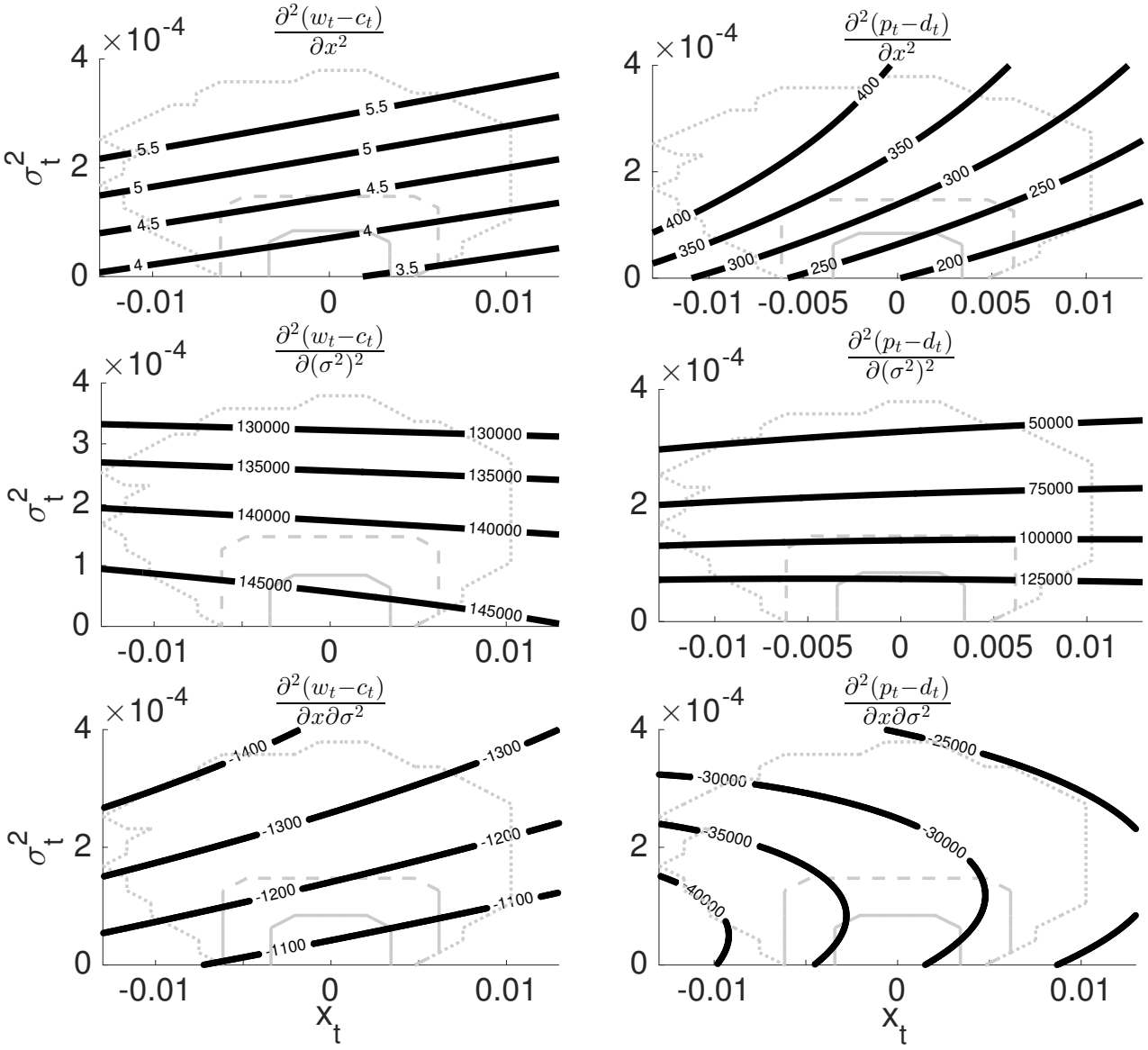
The graph shows isolines for the absolute errors in the first derivative of the log wealth-consumption ratio (left panel) and the log price-dividend ratio (right panel) with respect to the states x and σ^2 of the log-linearization (black solid lines). The (grey) dotted, dashed and solid lines mark the respective areas into which 100%, 90% and 50% of the observations from 10^6 simulated data points fall. The parameter values are from the calibration of Bansal, Kiku, and Yaron (2012a), see Table 2.

calibration. We observe in Figure 7 that the errors in the derivatives with respect to σ^2 are especially large, with errors up to 3000 for the price-dividend ratio. As mentioned above, the main purpose of the BKY (2012) calibration is to amplify the role of the stochastic volatility channel by increasing its persistence. But as demonstrated in the figures, this effect introduces large non-linearities to the model that cannot be captured by the log-linearization and hence causes large approximation errors. Figure 8 shows that the second derivatives in the model are substantially different from 0 (which is the value assumed by the log-linearization) and they are especially large (more than 10^5!) for the second derivative with respect to σ^2 which is another reason for the large approximation errors reported in Table 3.

In general, the figures show that the stochastic volatility channel highly influences the nonlinear aspects of the model. But is it only the stochastic volatility that matters? Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012) analyze the accuracy of several solution methods in a neoclassical growth model with Epstein-Zin preferences and stochastic volatility. They report that higher-order approximations are needed to capture the non-linearities of the model. Bansal, Kiku, and Yaron (2012b) report approximation errors for the long-run risks model in their estimation study by comparing the results of the log-linearization to the results obtained by the discretization method of Tauchen and Hussey (1991) (see Table A.1 of their paper). Unfortunately, for their exercise, they use a simplified version of their model that only features long-run risks (and no stochastic volatility). They find rather small approximation errors. But in the long-run risk model, there are two sources of non-linearities: the stochastic volatility channel and the long-run risk channel. Hence when solving the model, it is essential to understand whether and how the interplay of the two components drives the non-linearities.

To obtain such an understanding, we analyze the approximation errors implied by the log-linearization for each of the two state variables of the model separately. In particular we first fix the stochastic volatility to its long-run mean, $\sigma_t = \bar{\sigma}^2 \ \forall \ t$, and secondly we solve the model without long-run risk, $x_t = 0 \ \forall \ t$. Table 4 shows the corresponding errors in the unconditional mean and standard deviation of the log wealth-consumption and log price-dividend ratio for the two cases. We find that, in line with the test results from Bansal, Kiku, and Yaron (2012b), for the one-dimensional model with only long-run risks the approximation errors are very small with a maximum error of 0.21%). For the second case, without long-run risks and only stochastic volatility, the errors are slightly larger but still remain below 7.1%. However, for the full model with long-run risk and stochastic volatility approximation errors increase dramatically with a maximum error of 26.9% for the volatility of the log price-dividend ratio. This finding suggests that neither the stochastic volatility alone nor the long-run risks component alone introduces the non-linearities in the model; instead it is the simultaneous presence and interplay of the two features which makes the model so difficult to solve.

Figure 8: Approximation Errors in the Second Derivatives of the log Wealth-Consumption and Price-Dividend Ratio



The graph shows isolines for the absolute errors in the second derivative of the log wealth-consumption ratio (left panel) and the log price-dividend ratio (right panel) with respect to the states x and σ^2 of the log-linearization (black solid lines). The (grey) dotted, dashed and solid lines mark the respective areas into which 100%, 90% and 50% of the observations from 10^6 simulated data points fall. The parameter values are from the calibration of Bansal, Kiku, and Yaron (2012a), see Table 2.

Table 4: Approximation Errors for Each State of the Long-Run Risks Model Separately

	$E(w_t - c_t)$	$\sigma(w_t - c_t)$	$E(p_t - d_t)$	$\sigma(p_t - d_t)$
State: x_t	0.003%	0.024%	0.084%	0.21%
State: σ_t^2	0.14%	4.49%	2.62%	7.05%
Both States:	1.05%	12.25%	3.15%	26.90%

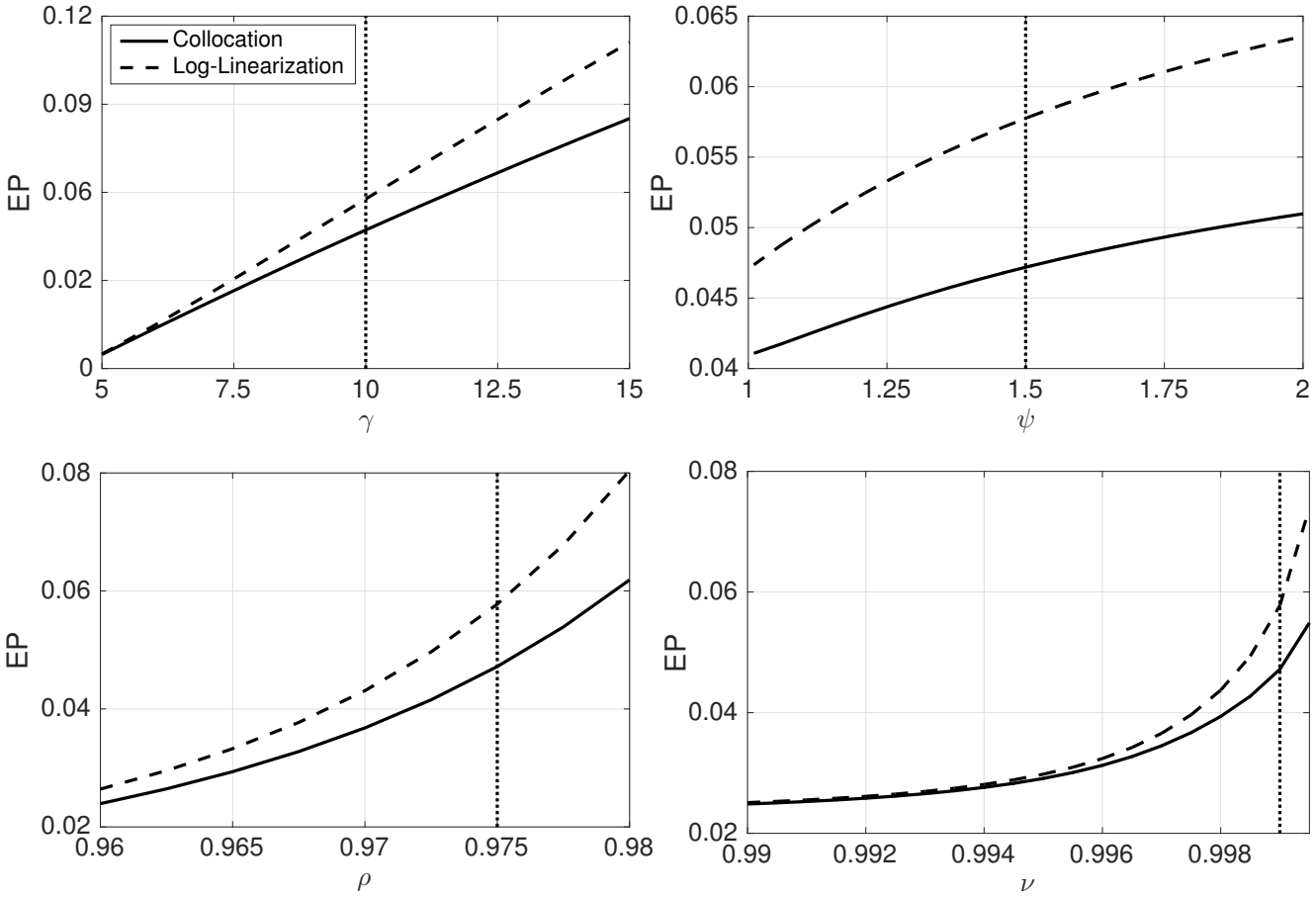
The table shows approximation errors in the unconditional mean and standard deviation of the log wealth-consumption and log price-dividend ratio induced by the log-linearization in the long-run risk model for each of the two state variables x_t and σ_t separately. For the case with only x_t, the state σ_t is simply set constant at its long-run mean $\bar{\sigma}^2$ (or equivalently $\nu = \sigma_w = 0$). For the case with only σ_t, x_t is set to 0 (or equivalently $\rho = \phi_x = \Phi = 0$). The parameter values are from the calibration of Bansal, Kiku, and Yaron (2012a), see Table 2.

4.4 Sensitivity of the Approximation Errors

As the previous results have shown, the non-linearities of the long-run risk model are highly dependent on its parameters. Therefore, in Figures 9 and 10 we analyze the approximation errors implied by the log-linearization with regard to changes in the parameters. In particular, we consider those parameters that are the main driving forces of the model, namely the risk aversion, γ, the intertemporal elasticity of substitution, ψ, the serial correlation in the long-run risk channel, ρ, and the stochastic volatility channel, ν.

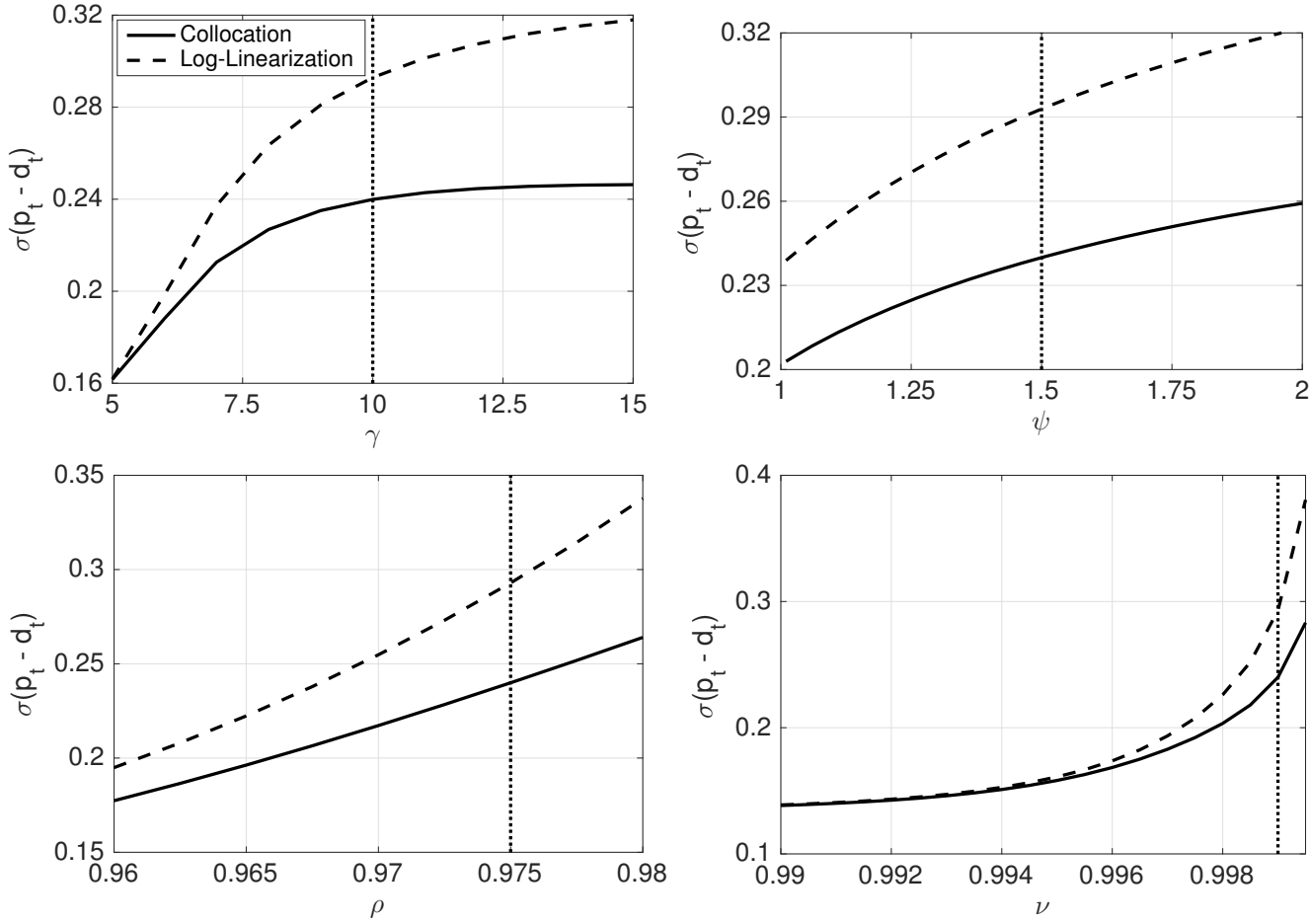
We find that, for this particular calibration, for a risk aversion of approximately 5, the log-linearized solution basically coincides with the solution from the projection approach, which suggests that a linear solution gives a reasonable approximation to the model. However, for this calibration also the implied model moments collapse with an equity premium below 1% and a sharp decrease in the volatility of the log price dividend ratio. When increasing the risk aversion the errors in the equity premium and the volatility of the log price-dividend ratio increase significantly, with a large overestimate of both quantities. Furthermore, in line with the previous results, the accuracy depends highly on the persistence of the processes for both the long-run risk and the stochastic volatility. We observe that even very small changes can dramatically increase approximation errors. For example, in the original calibration with a persistence in the long-run risk of $\rho = 0.975$ the overestimation of the equity premium is about 100 basis points (see Table 3). By slightly increasing ρ to 0.98, however, the difference doubles with an overestimation of 200 basis points. This very strong dependence on the persistence parameters also explains the large approximation errors in the estimation study of Schorfheide, Song, and Yaron (2014) (see Table 3) that finds a serial correlation in the long-run risk channel of $\rho = 0.993$. For the persistence in the conditional variance, ν, even a change of 0.0005, from 0.999 to 0.9995, increases the overestimation to 200 basis points. The figures also show that

Figure 9: Sensitivity of the Approximation Errors for the Equity Premium in the Long-Run Risks Model



The figure shows the equity premium obtained by the log-linearization (dashed line) as well as the premium obtained by the collocation projection (solid line) as a function of the model parameters γ, ψ, ρ and ν, respectively, assuming that the other parameters are kept constant. The results are computed for the calibration of Bansal, Kiku, and Yaron (2012a), see Table 2, and in each panel, the dotted vertical line denotes the estimate used in original calibration.

Figure 10: Sensitivity of the Approximation Errors for the Volatility of Price-Dividend Ratio in the Long-Run Risks Model



The figure shows the volatility of the log price-dividend ratio obtained by the log-linearization (dashed line) as well as the volatility obtained by the collocation projection (solid line) as a function of the model parameters γ, ψ, ρ and ν, respectively, assuming that the other parameters are kept constant. The results are computed for the calibration of Bansal, Kiku, and Yaron (2012a), see Table 2, and in each panel, the dotted vertical line denotes the estimate used in original calibration.

lowering the persistence parameters significantly decreases approximation errors. For example for $\nu = 0.99$ the approximation error becomes close to zero. However, for this calibration also the implied model moments collapse. Therefore it is especially important to pay attention to accurately solving the model as small changes to the parameters can have large impacts on the higher-order dynamics and hence introduce large approximation errors when using log-linear approximations; thus further applications of this class of models, require robust and accurate solution methods like the projection method presented in this paper.

5 Conclusion

We have investigated the existence of solutions for long-run risk models and the accuracy of the Campbell-Shiller log-linear approximation to those solutions. For existence, we have provided a relative existence result – if the model has a solution for CRRA preferences, then it has a solution for investors with a preference for an early resolution of uncertainty. Existence can be proven for the Bansal and Yaron (2004) model with CRRA preferences, so existence for early resolution follows.

To evaluate the quality of the log-linear solutions, we consider six recent models in the long-run risk literature: the original Bansal and Yaron (2004) model and the new calibration of Bansal, Kiku, and Yaron (2012a), the estimation of Schorfheide, Song, and Yaron (2014), the volatility-of-volatility model of Bollerslev, Xu, and Zhou (2015) and the work on real and nominal bonds of Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) and Bansal and Shaliastovich (2013). We find for very persistent underlying processes the approximation errors in log-linearization can be large and economically significant. For example, in the most recent calibration of the Bansal-Yaron long-run risk model (see Bansal, Kiku, and Yaron (2012a)), the approximation errors in the volatility of the log-price dividend ratio and the equity premium exceed 22% and become as large as 50% in the estimation study of Schorfheide, Song, and Yaron (2014). Models with lower persistence, such as the original Bansal and Yaron (2004) model or Bollerslev, Xu, and Zhou (2015), have much smaller approximation errors. The results for nominal bonds as in Bansal and Shaliastovich (2013) and Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) are particularly interesting – for the high level of persistence necessary to explain the equity premium, the log-linear approximation can actually produce an downward sloping yield curve, when the true yield curve is upward sloping.

Given the importance of long-run risk model in asset pricing, our results suggest that more sophisticated solution methods, such as projection methods, should be used when it comes to asset pricing models with highly persistent state processes.

Appendix

A Proofs for Section 3

Proof of Lemma 1. If $f \leq g^*$, then $Tf \leq Tg^* \leq Ug^* = g^*$. So, T maps $(0, g^*]$ into itself. By assumption, T is monotone. And so Tarski's (1955) Fixed-Point Theorem implies that T has a fixed point in the complete lattice $(0, g^*]$. □

Proof of Lemma 2. If $\theta < 1$, then by Jensen's inequality if $E(f|s)$ is finite, then $E(f^\theta|s)$ is finite. If $\theta > 1$, then again by Jensen's inequality if $E(f^\theta|s)$ is finite then $E(f|s)$ is finite. In both cases, T and U preserve $\mathcal{V}$.

(A) Let $0 \neq \theta \leq 1$. If $\theta > 0$, then x^θ is increasing in $x \in \mathbb{R}_{++}$. The monotonicity of the expected value operator implies for $f \leq g$ that

$$\begin{aligned} E(f^\theta|s) &\leq E(g^\theta|s), \\ \left[E(f^\theta|s)\right]^{1/\theta} &\leq \left[E(g^\theta|s)\right]^{1/\theta}, \end{aligned}$$

and thus $Tf \leq Tg$. If $\theta < 0$, then x^θ is decreasing in $x \in \mathbb{R}_{++}$. Now we obtain

$$\begin{aligned} E(f^\theta|s) &\geq E(g^\theta|s), \\ \left[E(f^\theta|s)\right]^{1/\theta} &\leq \left[E(g^\theta|s)\right]^{1/\theta}, \end{aligned}$$

and thus $Tf \leq Tg$ in this case as well. Trivially, $1/\theta \geq 1$ for $0 < \theta \leq 1$ and $1/\theta < 0$ for $\theta < 0$. In both cases, $x^{1/\theta}$ is convex for $x \in \mathbb{R}_{++}$. Therefore, by Jensen's inequality

$$E(X|s) = E((X^\theta)^{1/\theta}|s) \geq \left[E(X^\theta|s)\right]^{1/\theta}.$$

Thus, for $0 \neq \theta \leq 1$ it holds that $Tg \leq Ug$.

(B) Let $\theta > 1$. The monotonicity of the expected value operator implies for $f \leq g$ that $Uf \leq Ug$. Trivially, $1/\theta < 1$ and so $x^{1/\theta}$ is now concave for $x \in \mathbb{R}_{++}$. Therefore, by Jensen's inequality

$$E(X|s) = E((X^\theta)^{1/\theta}|s) \leq \left[E(X^\theta|s)\right]^{1/\theta}.$$

Hence, any pair $f, g \in \mathcal{V}$ with $f \leq g$ satisfies $Uf \leq Ug \leq Tg$.

□

Proof of Theorem 1.

(A) The asset-pricing model characterized by equations (1)–(6) has a solution for CRRA utility if and only if equation (8) has a solution $\hat{V}_t = \hat{V}_{t+1} = \hat{V}^*$ for $\theta = 1$. Lemma 3 implies that the equation also must have a solution for $0 \neq \theta < 1$, that is, for Epstein-Zin utility. Since λ and thus ψ are fixed, the conditions on γ follow.

(B) The asset-pricing model characterized by equations (1)–(6) has a solution for Epstein-Zin utility if and only if equation (8) has a solution $\hat{V}_t = \hat{V}_{t+1} = \hat{V}^*$ for $\theta \neq 1$. If $\theta > 1$, then Lemma 3 implies that the equation also must have a solution for $\theta = 1$, that is, for CRRA utility. Since λ and thus ψ are fixed, the conditions on γ follow.

□

Sketch of Proof of Theorem 2. The following lines are closely related to the work in de Groot (2015). In the online appendix de Groot (2015) shows how to derives closed-form solutions for the price-dividend ratio of the market portfolio in the model of Bansal and Yaron (2004) with CRRA preferences. He also presents a formal convergence theorem for the model without the long-run risk factor. Unfortunately he doesn't provide a convergence theorem for the pricing of the consumption claim for the full long-run risk model, which is needed for the existence of solutions for recursive preferences analyzed in this study. The following lines fill this gap.

Following Section B.4.2 in de Groot (2015) the solution of the wealth-consumption ratio $Z_{w,t} = \exp(z_{w,t})$ is of the form[16]

$$Z_{w,t} = \sum_{i=1}^{\infty} \delta^i \exp\left(A_{1,i} + A_{2,i}\bar{\sigma}^2 + A_{3,i}x_t + A_{4,i}\phi_x^2 + A_{5,i}(\bar{\sigma_t}^2 - \bar{\sigma}^2) + A_{6,i}\phi_\sigma^2\right).$$

The coefficients for the wealth-consumption ratio can be obtained in the same way as conducted by de Groot (2015) for the price-dividend ratio. To save space we do not state the full coefficients here. Please contact the authors for the detailed derivation of the coefficients. Define

$$Z_w^i \equiv \delta^i \exp\left(A_{1,i} + A_{2,i}\bar{\sigma}^2 + A_{3,i}x_t + A_{4,i}\phi_x^2 + A_{5,i}(\bar{\sigma_t}^2 - \bar{\sigma}^2) + A_{6,i}\phi_\sigma^2\right).$$

$Z_{w,t}$ is finite if

$$\lim_{i\to\infty} \left| \frac{Z_w^{i+1}}{Z_w^i} \right| < 1. \tag{20}$$

[16]Note that the notation is slightly different from the specification in de Groot (2015). While de Groot (2015) summarizes the constant terms $A_{2,i}$ and $A_{4,i}$ in one term called C_i^{BY} in the paper, we separate the two terms as the first term captures the influence of the short term shock to consumption growth, while the second term captures the influence of shock to the long-run growth rate.

Inserting the solutions for the coefficients we obtain

$$\begin{aligned}
\lim_{i\to\infty} A_{1,i+1} - A_{1,i} &= \mu_c \left(1 - \frac{1}{\psi}\right) \equiv B \\
\lim_{i\to\infty} A_{2,i+1} - A_{2,i} &= 0.5 \left(1 - \frac{1}{\psi}\right)^2 \equiv B_c \\
\lim_{i\to\infty} A_{3,i+1} - A_{3,i} &= 0 \\
\lim_{i\to\infty} A_{4,i+1} - A_{4,i} &= 0.5 \left(\frac{1 - \frac{1}{\psi}}{1-\rho}\right)^2 \bar{\sigma}^2 \equiv B_x \\
\lim_{i\to\infty} A_{5,i+1} - A_{5,i} &= 0 \\
\lim_{i\to\infty} A_{6,i+1} - A_{6,i} &= \frac{1}{8}\left(\left(\frac{\left(1-\frac{1}{\psi}\right)^2 \phi_x^2}{(1-\rho)(1-\nu)}\right)^2 + 2\left(\frac{\left(1-\frac{1}{\psi}\right)^2 \phi_x}{(1-\rho)(1-\nu)}\right)^2 + \left(\frac{\left(1-\frac{1}{\psi}\right)^2}{(1-\nu)}\right)^2\right) \equiv B_\sigma
\end{aligned}$$

Hence there exists a solution for the wealth-consumption ratio $Z_{w,t}$ if and only if

$$\delta \exp\left(B + B_c\bar{\sigma}^2 + B_x\phi_x^2 + B_\sigma\phi_\sigma^2\right) < 1. \tag{21}$$

□

B Computational Methods for Asset Pricing Models with Recursive Preferences

The common approach to solve for equilibrium dynamics is to log-linearize the model around its steady state. A discussion of log-linearization methods requires careful attention to several important differences among some well-known approaches. Standard log-linearization methods as in Judd (1996) or Collard and Juillard (2001) linearize around the deterministic steady state of the model. In a deterministic model, recursive preferences collapse to the case of CRRA preferences and hence the risk aversion has no influence (as there is no risk). But if the risk aversion has significant influence in the stochastic model, linearizing around the deterministic steady state might not be the best choice. Therefore new techniques have been developed that linearize around the risky steady state of the model (see, for example, Juillard (2011), de Groot (2013) or Meyer-Gohde (2014)).[17] Another drawback of the standard log-linearization is that the policies are independent of the volatility of the model (see Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012)). But as Bansal and Yaron

[17] These authors define the risky steady state as the state where, in absence of shocks in the current period, the agent decides to stay at the current state while expecting shocks in the future and knowing their probability distribution.

(2004) point out, stochastic volatility is one of the key features of the long-run risk model and essential for asset-pricing dynamics. Hence a log-linear approximation for asset-pricing models with recursive preferences and stochastic volatility must account for both features, the risk-adjustment of the steady state and the effects of volatility. Bansal and Yaron (2004) use a linearization technique based on the Campbell and Shiller (1988) return approximation that meets these requirements which, therefore, has been used extensively for solving asset-pricing models with recursive preferences (Segal, Shaliastovich, and Yaron (2015), Bansal, Kiku, and Yaron (2010), Bansal, Kiku, and Yaron (2012a), Bollerslev, Tauchen, and Zhou (2009), Kaltenbrunner and Lochstoer (2010), Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), Drechsler and Yaron (2011), Bansal and Shaliastovich (2013), Constantinides and Ghosh (2011), Bansal, Kiku, Shaliastovich, and Yaron (2014) or Beeler and Campbell (2012), among others).[18] One reason for its popularity is, that it allows for quasi-closed form solutions for many different model specifications, for example when shocks to the economy are normal. The log-linearization technique to solve asset pricing models with recursive preferences is described in Section B.1.

This study analyzes the log-linearized model solution with regard to existence properties and the influence of higher order dynamics on equilibrium outcomes that can, by construction, not be captured by the log-linear approximation described below. For CRRA preferences closed-form solutions for various model specifications can be computed. Unfortunately, for the case of recursive preferences, there are no such solutions. Therefore we need a highly accurate solution method which is capable to correctly capture higher-order features of the asset returns. A convenient choice are projection methods that allow to choose the approximation degree as well as the size of the approximation interval in order to be able to capture higher-order dynamics that are driven by the tails of the distribution. Projection methods are a general-purpose tool for solving functional equations. They were first introduced by physicists and engineers to solve partial differential equations, but they can be used to solve the types of fixed-point equations that arise in economics. (See Judd (1992) for an introduction or Chen, Cosimano, and Himonas (2014) for a brief overview how to apply projection methods to asset pricing models.) A detailed description of projection methods and how they can applied to solve the equilibrium conditions (2) and (6) is given in Appendix B.2.

[18] Another approach, proposed by Kogan and Uppal (2001) and used for example in Hansen, Heaton, Lee, and Roussanov (2007) and Hansen, Heaton, and Li (2008) is to linearize around the special case of unit elasticity of substitution $\psi = 1$ where the wealth-consumption ratio is constant. However most of the follow-up work in the long-run risk literature has focused on the log-linearization used in Bansal and Yaron (2004), which is why we concentrate on this approximation.

B.1 Log-Linearization Applied to Asset Pricing Models with Recursive Preferences

Here we provide a short sketch of the linearization method. For a general description of the method see Eraker (2008) and Eraker and Shaliastovich (2008). Assume that the log price-dividend ratio of asset i, $z_{i,t}$ is a linear function of the state variables

$$z_{i,t} = A_{0,i} + A_i y_t \tag{22}$$

where $y_t \in \mathbb{R}^l$ is the state vector describing the economy and $A_{0,i} \in \mathbb{R}^1$ and $A_i \in \mathbb{R}^l$ are the unknown linearization coefficients. The log return of the asset i, $r_{i,t+1}$ is then defined as

$$r_{i,t+1} = \log\left(e^{z_{i,t+1}} + 1\right) - z_{i,t} + \Delta d_{i,t+1} \tag{23}$$

where $\Delta d_{i,t+1}$ is the log growth rate of dividends. Making use of the Campbell and Shiller (1988) return approximation one gets

$$r_{i,t+1} \approx \kappa_{i,0} + \kappa_{i,1} z_{i,t+1} - z_{i,t} + \Delta d_{i,t+1} \tag{24}$$

with the linearizing constants

$$\kappa_{i,1} = \frac{e^{\bar{z}_i}}{1 + e^{\bar{z}_i}} \tag{25}$$

$$\kappa_{i,0} = -\log\left((1-\kappa_{i,1})^{1-\kappa_{i,1}} \kappa_{i,1}^{\kappa_{i,1}}\right) \tag{26}$$

that only depend on the model implied mean price-dividend ratio $\bar{z}_i = A_{0,i} + A_i E(y_t)$. Plugging the return approximation for the return on wealth (24) into the equilibrium condition (6) yields

$$E_t\left[e^{\theta \log\delta + (\theta - \frac{\theta}{\psi})\Delta c_{t+1} + \theta(\kappa_{w,0} + \kappa_{w,1} z_{w,t+1} - z_{w,t})}\right] = 1. \tag{27}$$

The equilibrium condition now only depends on the state of the economy and the linearization coefficients $A_{0,i}$ and A_i. As the equilibrium equation has to hold for any realization of the state of the economy, one can collect the terms for each state to obtain a square system of $l+1$ equations. Once we have solved for the return on wealth one can apply the linearization approach to the general pricing equation (2) to solve for the log price-dividend ratio of any asset i. For certain state processes the expectation can be evaluated analytically, as for example for processes with normal innovations as in Bansal and Yaron (2004) or Bollerslev, Tauchen, and Zhou (2009). This allows for quasi closed-form solutions for the linearization coefficients that only depend on the linearization constants $\kappa_{i,0}$ and $\kappa_{i,1}$. Eraker (2008), Eraker and Shaliastovich (2008) and Drechsler and Yaron (2011) show how to generalize the approach

to include general affine processes and jumps.

B.2 Projection Methods for Functional Equations

Projection methods (see Judd (1992) for an introduction or Chen, Cosimano, and Himonas (2014) for a brief overview) are a general tool to solve functional equations of the form

$$(\mathcal{G}z)(x) = 0, \tag{28}$$

where the variable x resides in a (state) space $X \subset \mathbb{R}^l$, $l \geq 1$, and z is an unknown solution function with domain X, so $z : X \to \mathbb{R}^m$. The given operator $\mathcal{G}$ is a continuous mapping between two function spaces. Note that solving equation (28) requires finding an element z in a function space–that is, in an infinite-dimensional vector space.

The first central step of a projection method is to approximate the unknown function z on its domain X by a linear combination of basis functions. For the applications in this paper, it suffices to assume that the domain X is bounded and that the basis functions are polynomials.[19] For a set $\{\Lambda_k\}_{k\in\{0,1,\ldots,n\}}$ of chosen basis functions the approximation $\hat{z}$ of z is

$$\hat{z}(x;\boldsymbol{\alpha}) = \sum_{k=0}^{n} \alpha_k \Lambda_k(x), \tag{29}$$

where $\boldsymbol{\alpha} = [\alpha_0, \alpha_1, \ldots, \alpha_n]$ are unknown coefficients. Replacing the function z in equation (28) by its approximation $\hat{z}$, we can define the residual function $\hat{F}(x;\boldsymbol{\alpha})$ as the error in the original equation,

$$\hat{F}(x;\boldsymbol{\alpha}) = (\mathcal{G}\hat{z})(x;\boldsymbol{\alpha}). \tag{30}$$

Instead of solving equation (28) for the unknown function z, we now attempt to choose coefficients $\boldsymbol{\alpha}$ to make the residual $\hat{F}(x;\boldsymbol{\alpha})$ zero. Note that instead of finding an element in an infinite-dimensional vector space we are now looking for a vector in $\mathbb{R}^{n+1}$. Obviously, this approximation step greatly simplifies the mathematical problem.

This problem is unlikely to have an exact solution, so the second central step of a projection method is to impose certain conditions on the residual function, the so-called "projection" conditions, to make the problem solvable. In other words, the purpose of the projection conditions is to establish a set of requirements that the coefficients $\boldsymbol{\alpha}$ must satisfy. For a formulation of the projection conditions, define a "weight function" (term) $w(x)$ and a set of "test" functions $\{g_k(x)\}_{k=0}^{n}$. We can then define an inner product between the residual

[19] In addition to polynomial approximations, approximations using cubic splines or B-splines are often very useful.

function $\hat{F}$ and the test function g_k,

$$\int_X \hat{F}(x;\boldsymbol{\alpha})g_k(x)w(x)dx.$$

This inner product induces a norm on the function space X. Natural restrictions for the coefficient vector α are now the projection conditions,

$$\int_X \hat{F}(x;\boldsymbol{\alpha})g_k(x)w(x)dx = 0,\ k = 0, 1, \ldots, n. \tag{31}$$

Observe that this system of equations imposes $n+1$ conditions on the $(n+1)$-dimensional vector α. Different projection methods vary in the choice of the weight function and the set of test functions. In this paper we describe two different projections, the collocation and the Galerkin method.

A collocation method chooses $n+1$ distinct nodes in the domain, $\{x_k\}_{k=0}^n$, and defines the test functions g_k by

$$g_k(x) = \begin{cases} 0 \text{ if } x \neq x_k \\ 1 \text{ if } x = x_k. \end{cases}$$

With a weight term $w(x) \equiv 1$, the projection conditions (31) simplify to

$$\hat{F}(x_k;\boldsymbol{\alpha}) = 0,\ k = 0, 1, \ldots, n. \tag{32}$$

Simply put, the collocation method determines the coefficients in the approximation (29) by solving the square system (32) of nonlinear equations.

The Galerkin method uses the fact that Chebyshev polynomials are orthogonal on $[-1, 1]$ with respect to the inner product using the weight function $w(x) \equiv \frac{1}{\sqrt{1-x^2}}$. Hence the Galerkin method uses the basis functions as the test functions, $g_k(x) = \Lambda_k(x)$ and the projection conditions (31) become

$$\int_X \hat{F}(x;\boldsymbol{\alpha})\Lambda_k(x)\frac{1}{\sqrt{1-x^2}}dx = 0,\ k = 0, 1, \ldots, n. \tag{33}$$

Next we show how to apply the general projection approach to solve the equilibrium pricing equations (6) and (2).

B.2.1 Projection Methods Applied to Asset-Pricing Models

To apply a projection method to the asset-pricing model, we express the equilibrium conditions as a functional equation of the type (28). For this purpose, we need to choose an appropriate state space and perform the usual transformation from an equilibrium described by infinite

sequences (with a time index t) to the equilibrium being described by functions of some state variables(s) x on a state space X. We denote the current state of the economy by x and the subsequent state in the next period by x'. (For example in the original model by Mehra and Prescott (1985), the state x is log consumption growth and $X \subset \mathbb{R}^1$; in the model of Bansal and Yaron (2004), the state x consists of the long-run mean of consumption growth (denoted by x_t in that paper) and the variance of consumption growth (denoted by σ_t^2), so $X \subset \mathbb{R}^2$.) We assume that the probability distribution of next period's state x' conditional on the current state x is defined by a density f_x.

First note that we solve the model in two steps. In the first step, we use the projection method to solve the wealth-Euler equation (6) to obtain the return on wealth. Once the return on wealth is known, then, in a second step, we can solve for any asset return by applying the projection approach to equation (2). For the first step, write equation (6) in state-space representation

$$E\left[\exp\left(\theta\log\delta - \frac{\theta}{\psi}\Delta c(x'|x) + \theta r_w(x'|x)\right)\middle| x\right] = 1, \quad \forall x, \tag{34}$$

where lower case letters denote logs of variables and $\Delta c(x'|x) = c(x') - c(x)$. We write the model in logs, because the function we solve for is the log wealth-consumption ratio $z_w(x) = \log\left(\frac{W(x)}{C(x)}\right)$. Next, write the state-dependent log return of the aggregate consumption claim as

$$\begin{aligned} r_w(x'|x) &= \log\left(\frac{W(x')}{W(x) - C(x)}\right) = \log\left(\frac{\frac{W(x')}{C(x')}}{\frac{W(x)}{C(x)} - 1} \times \frac{C(x')}{C(x)}\right) \\ &= z_w(x') - \log\left(e^{z_w(x)} - 1\right) + \Delta c(x'|x). \end{aligned} \tag{35}$$

Inserting the last term in equation (34) yields

$$E\left[\exp\left(\theta\left(\log\delta + (1 - \frac{1}{\psi})\Delta c(x'|x) + z_w(x') - \log\left(e^{z_w(x)} - 1\right)\right)\right) - 1\middle| x\right] = 0, \quad \forall x. \tag{36}$$

Equivalently,

$$0 = \int_X \left[\exp\left(\theta\left(\log\delta + (1 - \frac{1}{\psi})\Delta c(x'|x) + z_w(x') - \log\left(e^{z_w(x)} - 1\right)\right)\right) - 1\right] \mathrm{d}f_x \tag{37}$$

which is a functional equation of the form (28) and allows us to apply the projection approach.

The unknown solution function to this equilibrium condition, z_w, is an element of a function space which is an infinite-dimensional vector space. A key feature of every projection method is to approximate the solution function z_w by an element from a finite-dimensional space.

Specifically, we use the approximation $\hat{z}_w(x;\boldsymbol{\alpha_w}) = \sum_{k=0}^{n} \alpha_{w,k}\Lambda_k(x)$, where $\{\Lambda_k\}_{k\in\{0,1,\dots,n\}}$ is a set of chosen (known) basis functions and $\boldsymbol{\alpha_w} = [\alpha_{w,0}, \alpha_{w,1}, \dots, \alpha_{w,n}]$ are unknown coefficients. Replacing the exact solution $z_w(x)$ by the approximation $\hat{z}_w(x;\boldsymbol{\alpha_w})$ leads us to the residual function $\hat{F}_w$ for the rearranged wealth-Euler equation (37), which is defined by

$$\hat{F}_w(x;\boldsymbol{\alpha_w}) = \int_X \left[\exp\left(\theta\left(\log\delta + (1-\frac{1}{\psi})\Delta c(x'|x) + \hat{z}_w(x') - \log\left(e^{\hat{z}_w(x)}-1\right)\right)\right) - 1\right] \mathrm{d}f_x. \tag{38}$$

We can determine values for the unknown solution coefficients $\boldsymbol{\alpha_w}$ by imposing a projection condition on the residual term $\hat{F}_w(x;\boldsymbol{\alpha_w})$. In this paper we employ two different such projection conditions, the collocation and the Galerkin method, see Appendix B.2. The values for the coefficients $\boldsymbol{\alpha_w}$ determine the state-dependent wealth-consumption ratio $\hat{z}_w(x;\boldsymbol{\alpha_w})$ which in turn leads to the (approximate) return function of the aggregate consumption claim, $\hat{r}_w(x'|x;\boldsymbol{\alpha_w}) = \hat{z}_w(x';\boldsymbol{\alpha_w}) - \log\left(e^{\hat{z}_w(x;\boldsymbol{\alpha_w})}-1\right) + \Delta c(x'|x)$.

With $\hat{r}_w(x'|x;\boldsymbol{\alpha_w})$ at hand, we can now develop an approach to compute the return of any asset i using equation (2). Analogous to the first step, we solve for the log price-dividend ratio $z_i(x) = \log\left(\frac{P(x)}{D(x)}\right)$ and rewrite the state-dependent log return of asset i as

$$\begin{aligned} r_i(x'|x) &= \log\left(\frac{P_i(x')+D_i(x')}{P_i(x)}\right) = \log\left(\frac{\frac{P_i(x')}{D_i(x')}+1}{\frac{P_i(x)}{D_i(x)}} \times \frac{D_i(x')}{D_i(x)}\right) \\ &= \log\left(e^{z_i(x')}+1\right) - z_i(x) + \Delta d_i(x'|x). \end{aligned} \tag{39}$$

Writing the Euler equation (2) in state-space representation and formulating it in logs yields

$$E\left[\exp\left(\theta\log\delta - \frac{\theta}{\psi}\Delta c(x'|x) + (\theta-1)r_w(x'|x) + r_i(x'|x)\right)\middle| x\right] = 1. \tag{40}$$

Substituting the return expressions (35) and (39) into this equations and replacing the log price-dividend ratio $z_i(x) = p_i(x) - d_i(x)$ by its approximation $\hat{z}_i(x;\boldsymbol{\alpha_i}) = \sum_{k=0}^{n}\alpha_{i,k}\Lambda_k(x)$ leads to the residual function

$$\begin{aligned} \hat{F}_i(x;\boldsymbol{\alpha_i}) = \int_X \Bigg[\exp\Bigg(&\theta\log\delta - \frac{\theta}{\psi}\Delta c(x'|x) + (\theta-1)\hat{r}_w(x'|x;\boldsymbol{\alpha_w}) \\ &+ \log\left(e^{\hat{z}_i(x';\boldsymbol{\alpha_i})}+1\right) - \hat{z}_i(x;\boldsymbol{\alpha_i}) + \Delta d_i(x'|x)\Bigg) - 1\Bigg]\mathrm{d}f_x \end{aligned} \tag{41}$$

Recall that the coefficients $\boldsymbol{\alpha_w}$ and thus the function $\hat{r}_w(x'|x;\boldsymbol{\alpha_w})$ have been computed previously. Therefore, we can now apply one of the projection conditions to solve for the unknown vector $\boldsymbol{\alpha_i}$.

In sum, we apply the projection method twice. In the first step, we approximate the

log wealth-consumption ratio $\hat{z}_w(x;\boldsymbol{\alpha_w})$ by applying the projections on the residual function of the wealth-Euler equation (38). Once $\boldsymbol{\alpha_w}$ is known, the projections can be applied to equation (41) to solve for the price-dividend ratio $\hat{z}_i(x;\boldsymbol{\alpha_i})$ of any asset i. Formally, the algorithm can be described as follows.

Algorithm *Solving Asset-Pricing Models with Recursive Preferences.*

Initialization. *Define the state space $X \subset \mathbb{R}^l$; choose the functional forms for $\hat{z}_w(x;\boldsymbol{\alpha_w})$ and $\hat{z}_i(x;\boldsymbol{\alpha_i})$ as well as the projection method.*

Step 1. *Use the wealth-Euler equation* (6) *together with the approximated log wealth-consumption ratio $\hat{z}_w(x;\boldsymbol{\alpha_w})$ and the definition of the return equation* (35) *to derive the residual function for the return on wealth*

$$\hat{F}_w(x;\boldsymbol{\alpha_w}) = \int_X \left[\exp\left(\theta\left(\log\delta + (1-\frac{1}{\psi})\Delta c(x'|x) + \hat{z}_w(x') - \log\left(e^{\hat{z}_w(x)}-1\right)\right)\right) - 1\right] \mathrm{d}f_x.$$

Compute the unknown solution coefficients $\boldsymbol{\alpha_w}$ by imposing the projections on $\hat{F}_w(x;\boldsymbol{\alpha_w})$.

Step 2. *Use the solution for the wealth-consumption ratio $\hat{z}_w(x;\boldsymbol{\alpha_w})$ and the Euler equation* (2) *for asset i together with the approximated log price-dividend ratio $\hat{z}_i(x;\boldsymbol{\alpha_i})$ and the definition of the return equation* (39) *to derive the residual function for asset i,*

$$\hat{F}_i(x;\boldsymbol{\alpha_i}) = \int_X \Bigg[\exp\Bigg(\theta\log\delta - \frac{\theta}{\psi}\Delta c(x'|x) + (\theta-1)\hat{r}_w(x'|x;\boldsymbol{\alpha_w}) + \log\left(e^{\hat{z}_i(x';\boldsymbol{\alpha_i})}+1\right) - \hat{z}_i(x;\boldsymbol{\alpha_i}) + \Delta d_i(x'|x)\Bigg) - 1\Bigg]\mathrm{d}f_x$$

Compute the unknown solution coefficients $\boldsymbol{\alpha_i}$ by imposing the projections on $\hat{F}_i(x;\boldsymbol{\alpha_i})$.

Evaluation. *Choose a set of evaluation nodes $\mathbb{X}^e = \{x_j^e : 1 \leq j \leq m^e\} \subset X$ and compute approximation errors in the residual function of the wealth portfolio and the residual function of asset i. If the errors do not satisfy a predefined error bound, start over at Initialization and change the number of approximation nodes or the degree of the basis functions.*

To actually implement the algorithm, we need to specify additional algorithmic details such as the choices for basis functions and the integration technique.

B.2.2 Algorithmic Ingredients

In the **Initialization** step, we need to choose a set of basis functions for the polynomial approximation, a projection method and a set of nodes. To simplify the presentation, we describe the necessary choices for a one-dimensional state space approximated over an interval $X = [x_{min}, x_{max}]$. We approximate the solution functions z_w and z_i by Chebyshev polynomials (of the first kind), see Judd (1998). We obtain the Chebyshev polynomials via the recursive relationship

$$T_0(\xi) = 1, \quad T_1(\xi) = \xi, \quad T_{k+1}(\xi) = 2\xi T_k(\xi) - T_{k-1}(\xi),$$

with $T_k : [-1, 1] \to \mathbb{R}$. Since we need to approximate functions on the domain X and the Chebyshev polynomials are defined on the interval $[-1, 1]$, we need to transform the argument for the polynomials. The basis functions for the approximate solutions $\hat{z}_w(x; \boldsymbol{\alpha_w})$ and $\hat{z}_i(x; \boldsymbol{\alpha_i})$ are given by

$$\Lambda_k(x) = T_k\left(2\left(\frac{x - x_{min}}{x_{max} - x_{min}}\right) - 1\right) \tag{42}$$

for $k = 0, 1, \ldots, n$.

In this paper we only show the results using the collocation method but we verified the solutions using the Galerkin approach. The application of a projection method requires a set of nodes, $\mathbb{X} = \{x_j : 0 \leq j \leq m\} \subset X$; we choose the $m+1$ zeros of the Chebyshev polynomial T_{m+1}. These points are called Chebyshev nodes,

$$\xi_j = \cos\left(\frac{2j+1}{2m+2}\pi\right), \qquad j = 0, 1, \ldots, m.$$

Since all Chebyshev nodes are in the interval $[-1, 1]$, we need to transform them to obtain nodes in the state space X. This transformation is

$$x_j = x_{min} + \frac{x_{max} - x_{min}}{2}(1 + \xi_j), \qquad j = 0, 1, \ldots, m.$$

For the collocation method, the number of basis functions, $n + 1$, must be identical to the number of approximation nodes, $m+1$, and so $m = n$. In **Step 1** (and **Step 2**, if applicable), we must solve the projection conditions involving the residual function. The residual functions defined in equations (38) and (41) contain a conditional expectations operator, which also requires numerical calculations. The underlying exogenous processes in the models we consider are normally distributed, and so we apply Gauss-Hermite quadrature to calculate expectations.

The collocation approach leads to a square system of nonlinear equations, see Appendix B.2, which can be solved with a standard nonlinear equation solver. The Galerkin projection is slightly more complex, and uses integral operators as projection conditions; these in turn can

be accurately approximated by Gauss-Chebyshev quadrature.

For the **Evaluation** step we use $m^e >> m$ equally spaced evaluation nodes in X to evaluate the errors in the residual function. In particular, for asset i we compute the root mean squared errors (RMSE) and maximum absolute errors (MAE) in the residual function (41); these errors are

$$\text{RMSE}_i = \sqrt{\frac{1}{m^e}\sum_{j=1}^{m^e} \hat{F}_i(x_j^e|\boldsymbol{\alpha_i})^2}, \tag{43}$$

$$\text{MAE}_i = \max_{j=1,2,\ldots,m^e} |\hat{F}_i(x_j^e|\boldsymbol{\alpha_i})|, \tag{44}$$

respectively, with

$$x_j^e = x_{min} + \frac{x_{max} - x_{min}}{m^e - 1}(j-1), \qquad j = 1, \ldots, m^e. \tag{45}$$

C Accuracy of the Projection Method

Table 5 demonstrates the accuracy of the projection approach. We consider the long-run risk model of Bansal and Yaron (2004) with constant volatility where there exist closed form solutions for the case of CRRA preferences. In the case of recursive preferences we determine the accurate solution using the projection approach with a very large degree and state space. (We use $n_\sigma = 50$ and increase the degree until the highest order coefficient is close to zero. We double check the solution by using the discretization method of Tauchen and Hussey (1991) with a very large number of discretization nodes). We find that for the calibration with $\rho = 0.95$ already a first order approximation with an approximation interval of $n_\sigma = 1$ standard deviation around the unconditional mean of x_t provides a very accurate solution with an approximation error of 1.51e-5 for the case with recursive utility and $\gamma = 10$. For the high persistence case with $\rho = 0.99$ a larger degree is required and the degree four polynomial is sufficient to compute a highly accurate solution. Overall we observe, that the projection method provides highly accurate solutions for all specifications considered in this example.

We report the results when solving the Euler equation for wealth. Alternatively we could solve the fixed-point equation for utility. The results this way are almost identical – the coefficients differ by less than 10^{-12}.

D The Volatility of Volatility Factor

This section analyzes how log-linearization affects model outcomes when the model dynamics are described by a square-root process as for example in Bollerslev, Tauchen, and Zhou (2009),

Table 5: Accuracy of the Projection Method

	Closed-F.	**Log-Lin**	**Projection**			**Discretization**		
			$n=1$ $n_\sigma=1$	$n=4$ $n_\sigma=4$	$n=16$ $n_\sigma=32$	$n_D=5$	$n_D=10$	$n_D=50$
$\psi=1.5, \gamma=1/\psi$								
$\rho=0.95$								
$E(\frac{W}{C})$	1681.20	1681.16	1681.18	1681.20	1681.20	1669.99	1670.75	1671.00
Error	0	2.11e-5	1.19e-5	2.61e-8	2.60e-8	0.0067	0.0062	0.0060
$\rho=0.99$								
$E(\frac{W}{C})$	1868.36	1862.93	1865.54	1868.36	1868.36	3404.73	2121.64	1852.27
Error	0	0.0029	0.0015	1.21e-7	7.65e-11	0.8223	0.1356	0.0086
$\psi=1.5, \gamma=10$								
$\rho=0.95$								
$E(\frac{W}{C})$	-	1314.39	1314.59	1314.61	1314.61	1532.25	1514.25	1508.08
Error	-	1.66e-4	1.51e-5	4.37e-11	2.12e-12	0.1655	0.1518	0.1472
$\rho=0.99$								
$E(\frac{W}{C})$	-	517.13	518.97	529.39	529.39	869.23	653.99	570.65
Error	-	0.0231	0.0196	1.43e-9	4.91e-11	0.6419	0.2353	0.0779

The table shows the mean wealth-consumption ratio for the long-run risk model of Bansal and Yaron (2004) with constant volatility (Equations (7) with $\sigma_{c,t} = \sigma_{x,t} = \bar{\sigma}$ and $\eta_{c,t+1}$, $\eta_{x,t+1}$ i.i.d. normal.). Results are shown for the log-linearization, the projection as well as the discretization by Tauchen and Hussey (1991) with the extension of Floden (2007) that performs better for highly persistent processes. For the projection method solutions with three different degrees n where the approximation interval is set up n_σ standard deviations around the unconditional mean of the long-run risk process x_t are provided. For the discretization results are shown for three different numbers of approximation nodes n_D. The table also shows the relative error of the solutions, where in the case of $\gamma = 1/\psi$ the closed form solution is taken from de Groot (2015) and in the case of $\gamma \neq 1/\psi$ we compute the accurate solution by solving the model using the discretization method with a very large number of discretization nodes or equivalently the projection with a very large degree and state space. We use the same calibration as in Section 3.3 with $\delta = 0.9989$, $\mu_c = 0.0015$, $\bar{\sigma}^2 = 0.0078^2$ and $\phi_x = 0.044$.

Tauchen (2011) or Bollerslev, Xu, and Zhou (2015). For this purpose we use the parsimonious model formulation as in Bollerslev, Tauchen, and Zhou (2009) who take the basic model setup (7) without the long-run risk factor $\phi_x = 0$ and add vol-of vol modeled by a square root process q_t:

$$\begin{aligned} \sigma^2_{t+1} &= \bar{\sigma}^2(1-\nu) + \nu\sigma^2_t + \sqrt{q_t}\eta_{\sigma,t+1} \\ q_{t+1} &= \mu_q(1-\rho_q) + \rho_q q_t + \phi_q\sqrt{q_t}\eta_{q,t+1} \\ & \quad \eta_{\sigma,t+1}, \eta_{q,t+1} \sim i.i.d.\ N(0,1). \end{aligned} \tag{46}$$

As Tauchen (2011) notes, care is needed, as q_t can become negative in simulations if the volatility is too large compared to the mean of the process. The common approach in the literature is to assume a reflecting barrier at zero by replacing negative values with very small positive values to ensures positivity of the process (this approach has also been used for the stochastic volatility process in the original Bansal and Yaron (2004) study and many following papers). However, to compute model solutions, the assumption of a non-truncated distribution for the log-linearization is commonly used.

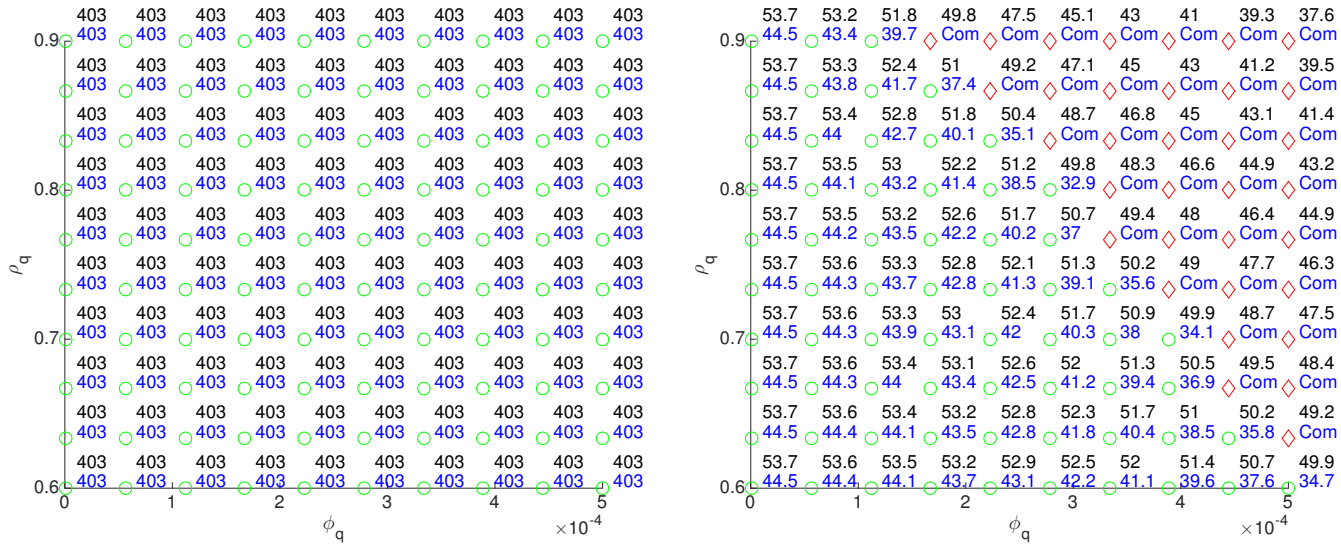
Take, for example, the calibration of Bollerslev, Tauchen, and Zhou (2009) given by $\delta = 0.997, \gamma = 10, \psi = 1.5, \mu_c = 0.0015, \nu = 0.978, \bar{\sigma}^2 = 0.0078^2$ and $\mu_q =$ 1e-6. Figure 11 shows model outcomes for CRRA preferences with $\psi = 1.5$ (Panel (a)) and the corresponding EZ case with $\gamma = 10$ (Panel (b)) for various persistence and volatility parameters of the vol-of-vol process ρ_q and ϕ_q. The black numbers show the true mean wealth-consumption ratio under the assumption of a reflecting boundary for q_t at zero. Blue values are the results from log-linearization under the assumption of a standard non-truncated normal distribution. Green circles denote convergence of both, the projection and the log-linearization approach. Red diamonds denote cases in which the log-linearization yields a complex solution, while the model solution using a truncated normal distribution is real. We find that, depending on the risk aversion, using the standard log-linearization technique can lead to complex solutions. This is for example the case for the calibration in Bollerslev, Tauchen, and Zhou (2009) with $\rho_q = 0.8$ and $\phi_q =$ 1e-3.[20]

So what are the determinants of the complexity of the linearized solution? The square-root specification of q_t implies that the coefficient for q_t is determined by a quadratic equation and hence may have more than one solution. The log-linear approximation of the log wealth-

[20] Bollerslev, Tauchen, and Zhou (2009) provide a real solution by assuming a fixed value for the linearization constant $\kappa = 0.9$. However this approach doesn't give a solution to the model but ex ante fixes the mean value of the price dividend ratio and hence significantly biases the model outcome.

Figure 11: Sensitivity Analysis and Existence Results in the Vol-of-Vol Model

(a) $\psi = 1.5, \gamma = 2/3$ (b) $\psi = 1.5, \gamma = 10$



The graph shows the convergence properties as well as the mean wealth-consumption ratio for the vol-of-vol model of Bollerslev, Tauchen, and Zhou (2009). The results are reported for a range of persistence parameters ρ_q and volatility parameters ϕ_q. Panel (a) depicts the case of CRRA utility with $\psi = 1.5$, while panel (b) depicts the corresponding cases with recursive utility and $\gamma = 10$. Black numbers show the mean wealth-consumption ratio obtained by the projection approach using a reflecting barrier at zero and blue numbers show the values obtained by the standard log-linearization with normal shocks. Green circles denote convergence of both, the projection and the log-linearization approach. Red diamonds denote cases in which the log-linearization yields a complex solution, while the model solution using a truncated normal distribution is real. The model parameters are given by $\delta = 0.997, \mu_c = 0.0015, \nu = 0.978, \bar{\sigma}^2 = 0.0078^2$ and $\mu_q =$ 1e-6.

consumption ratio $z_{w,t}$ has the following form

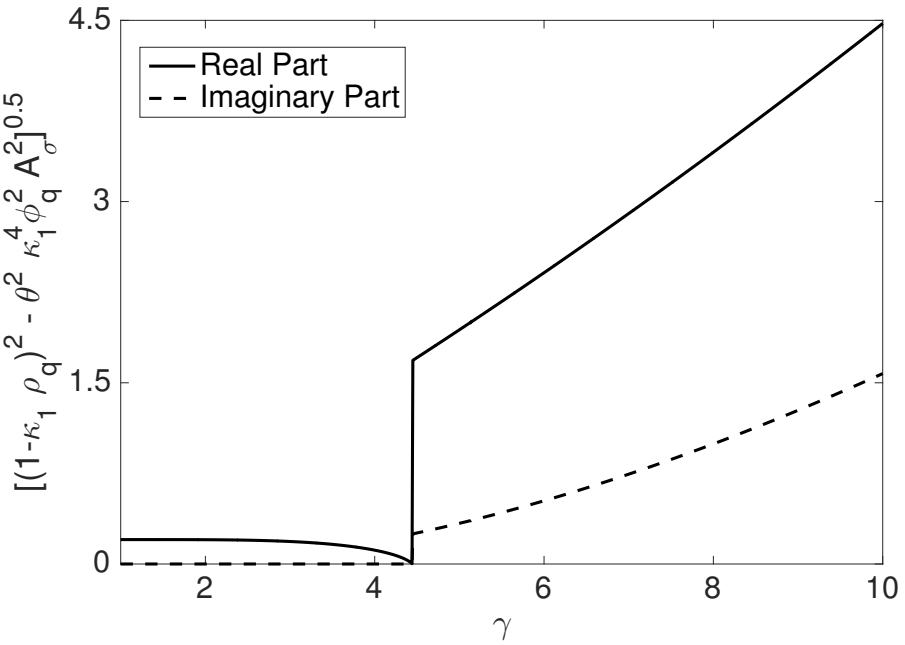
$$z_{w,t} = A_0 + A_\sigma \sigma_t^2 + A_q q_t \tag{47}$$

with the linearization coefficients (see Appendix B.1 for the derivation) given by

$$\begin{aligned} A_\sigma &= \frac{(1-\gamma)^2}{2\theta(1-k_1\nu)} \\ A_0 &= \frac{\log\delta + (1-\frac{1}{\psi})\mu_c + k_0 + k_1\Big[A_\sigma\bar{\sigma}^2(1-\nu) + A_q\mu_q(1-\rho_q)\Big]}{(1-k_1)} \\ A_q &= \frac{1-k_1\rho_q \pm \sqrt{(1-k_1\rho_q)^2 - \theta^2 k_1^4\phi_q^2 A_\sigma^2}}{\theta k_1^2\phi_q^2} \end{aligned} \tag{48}$$

We find that the coefficient for the vol-of-vol factor A_q has indeed two solutions. As Bollerslev, Tauchen, and Zhou (2009) show in their paper by the no arbitrage argument, the minus term is the economically meaningful root and the positive solution can be neglected. Complexity of the solution is determined by the term inside the square root in equation (48) given by $(1-k_1\rho_q)^2 - \theta^2 k_1^4\phi_q^2 A_\sigma^2$. So how does this term depend on the model parameters? Figure 12 shows the values of the square root term as a function of the risk aversion γ. In line with

Figure 12: Analysis of Square-Root Term in the Vol-of-Vol Model



The graph shows the real and complex part of the square root term that determines A_q as a function of the risk aversion γ for the vol-of-vol model of Bollerslev, Tauchen, and Zhou (2009). The model parameters are given by $\delta = 0.997, \mu_c = 0.0015, \nu = 0.978, \bar{\sigma}^2 = 0.0078^2, \mu_q =$ 1e-6, $\rho_q = 0.8$ and $\phi_q = 1 \times 10^{-3}$.

the results above, we find that for small γ the solution is well behaved with only a real and no imaginary part. However if we increase γ, θ becomes significantly larger (it goes from -3 for $\gamma = 2$ to -27 for $\gamma = 10$) and hence the real part of the term decreases. For a certain threshold (about 4.4 in this example) the term hits zero and the solution thereafter consists of a significant imaginary part. Also Panel (b) in Figure 11 shows, that the larger the persistence or the larger the volatility of the vol-of-vol process solutions become complex. Summarizing, using standard log-linearization with normal shocks to solve models with a large risk aversion and a persistent square-root process can yield complex solutions, even if real solutions under the assumption of a reflecting barrier exist. Hence when solving such models, either log-linearization with the assumption of a truncated normal distribution or more sophisticated methods like the projection approach described in this paper should be used.

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