Rational Addictive Behavior under Uncertainty[1]

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Abstract: We develop a new model of addictive behavior that takes as a starting point the classic rational addiction model of Becker and Murphy, but incorporates uncertainty. We model uncertainty through the Wiener stochastic process. This process captures both random events such as anxiety, tensions and environmental cues which can precipitate and exacerbate addictions, and those sober and thought-provoking episodes that discourage addictions. We derive closed-form expressions for optimal (and expected optimal) addictive consumption and capital trajectories and examine their global and local properties. Our theory provides plausible explanations of several important patterns of addictive behavior, and has novel implications for addiction control policy.

Keywords: Rational Addiction; Stochastic Control; Uncertainty

JEL classification: C61, D01 D11, I10, I18, K32

1 Introduction

Addiction to certain substances such as alcohol, tobacco, cocaine, marijuana, and heroin, or activities like gambling, eating, sex, watching television, playing computer games, and internet use, can be so powerful that it often has a dire effect on the lives of people involved. According to the 2011 and 2012 World Health Organization reports on the harmful use of alcohol and tobacco, these two addictive substances cause approximately 2.5 million and 5 million deaths respectively each year around the world. The number of people who are

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dependent on alcohol, tobacco or other substances far exceeds the death toll. Alcohol use is the world's third biggest risk factor for disease and disability and is a causal factor in 60 types of diseases and injuries and a component cause in 200 others. It is also closely associated with many serious social problems such as violence, child neglect and abuse, and absenteeism in the workplace. Tobacco smoking alone kills even more than acquired immune deficiency syndrome/human immunodeficiency virus (AIDS/HIV), malaria and tuberculosis combined. Across the globe, 12% of all deaths amongst people aged 30 years and above have been identified to be attributable to tobacco. In particular, tobacco smoking accounts for 71% of all lung cancer deaths and 42% of all chronic obstructive pulmonary disease. Both alcohol drinking and tobacco smoking contribute to family poverty whereby money spent on them can take away a significant part of total household income that may be necessary for the family's use of other goods and services. A worrisome development is that the consumption of alcohol and tobacco in developing countries is accelerating and more and more women have also begun to smoke and drink as these countries become more affluent and women also have more economic power. This trend is blindingly obvious in China.[4]

Addiction has long been recognized as a fundamental and baffling problem for researchers, social workers and governments.[5] The primary goal of studying (harmful) addiction is to try to understand its behavior and ultimately to explore its treatment and methods of prevention. Among many theories and models developed so far stands out Becker and Murphy (1988)'s theory of rational addiction (see e.g., Vuchinich and Heather 2003, West 2006, and Moss and Dyer 2010) as a classic economic analysis of addiction. Addictive behavior for drinking, smoking, eating, gambling and others is habit-forming, cannot be static and must be dynamic. The key symptoms of such behavior are tolerance, withdrawal and reinforcement. Tolerance means that repeatedly using a substance or doing an activity over time requires more and more of the substance or activity to achieve the same level of satisfaction as the individual previously experienced. Withdrawal is a negative state at which an individual will feel extremely uncomfortable when he reduces or stops the consumption of a substance or an activity. For instance, typical alcohol with-

[4]Although the number of smokers has been decreasing considerably in developed countries in the last few decades due to governments' legislations on cigarettes and unrelenting campaigns of concerned health groups against tobacco, smoking is still a very serious health problem in some developed countries. A recent BBC investigation by Peter Taylor (2014) states: "Though the (smoking) habit is slowly declining, about one in five British adults still smoke. Smoking among 20 to 34-year-odds has actually increased in the past few years. And though the tobacco industry insists it does not target children, every year 200,000 of those aged 11-15 start smoking."

[5]Surprisingly or not, even several academic journals including *Addiction*, *Addictive Behaviors*, *The American Journal on Addictions*, *Psychology of Addictive Behaviors*, are entirely devoted to the study of addiction. The sheer volume of research illuminates the immense importance of the subject.

drawal symptoms include agitation, delirium tremens, and seizures. Reinforcement refers to the type of behavior that the more an individual consumes a substance or partakes of an activity today, the more he wants to do in the future.[6] Becker and Murphy lay out a dynamic economic model that not only captures these fundamental features of addiction, but also gives sensible predictions about addictive behavior.

In their model, Becker and Murphy use a utility function to quantify the "benefit" or "pleasure" from consuming an addictive good and a normal good. The utility function depends both on the consumption level of the addictive good and the normal good and on the addictive capital that has been accumulated so far. They impose first- and second-order conditions on the utility function so that tolerance, withdrawal and reinforcement are satisfied. The individual is assumed to be fully aware of the negative consequence of consuming the addictive good and capable of weighing all options rationally and making a consistent plan to maximize utility over his life time. Becker and Murphy use a quadratic function to approximate the utility function and provide detailed analysis of the dynamic aspects of addictive consumption. Their modeling of rational addiction has subsequently become the standard approach to the study of consumption of addictive goods such as tobacco, alcohol, cocaine, coffee and gambling and has also found strong and consistent empirical support from a number of studies (see e.g., Becker, Grossman and Murphy 1991, 1994, Chaloupka 1991, Mobilia 1990, Keeler, Hu, Barnett and Manning 1993, Olekalns and Bardsley 1996, Grossman and Chaloupka 1998, Grossman, Chaloupka, and Sirtalan 1998, Suranovic, Goldfarb and Leonard 1999, Fenn, Antonovitz and Schroeter 2001, and Adda and Cornaglia 2006).

The purpose of the current paper is to develop a new model of addictive behavior that takes as its starting point the classic model of Becker and Murphy (1988), but incorporates a crucial factor –uncertainty– into the model. Uncertainty is a fact of life and even more so for addictive people. It is well-known that anxiety, insecurity and tension can precipitate and worsen an addiction. This mental state is often caused by the interactions of genetic factors with unpredictable and stressful events such as unemployment, death of a loved one, marital breakup, and peer pressure. Also random events come from exposure to environmental cues. As cases in point, sight of a cigarette advertisement can induce an irresistible impulse to smoke; warm and sunny weather boosts the mood for sex, while cold and gloomy weather dampens the mood for sex. Another unfortunate fact is that addicts of harmful substances usually make their lives more unpredictable, because they can lose their jobs more easily, their marriage can be less stable, and they are also more vulnerable to diseases and accidents. Generally speaking, uncertainty exacerbates addiction; addiction reinforces

[6] Becker and Murphy (1993) develop an advertising model which also shares this feature of reinforcement, that is, advertisements are complementary to consumption of the advertised good.

uncertainty. Vicious circles can arise in an addict's life. Events that can precipitate and aggravate addiction will be called harmful events, while events such as compelling campaigns against drugs that can discourage addiction will be called beneficial events. Our model accommodates both harmful and beneficial random events. We adopt the standard Wiener stochastic process to capture uncertainty; see e.g., Mirrlees (1963), Merton (1969, 1971), and Black and Scholes (1973). More precisely, in contrast to Becker and Murphy's deterministic addiction capital accumulation process, we use the Wiener stochastic process to describe how uncertainty influences the accumulation of addiction capital. In general we show in Proposition 2 that addictive consumption will decrease as the price of the addictive good increases. The incorporation of uncertainty entails simplification of some parts of the model. Our analysis therefore focuses on the essential and tractable case where the normal good is ignored or its consumption level is fixed. Surprisingly in this case closed-form solutions can be obtained for optimal addictive consumption and capital trajectories (Theorem 1). The explicit formulas reveal the complex evolutionary nature of addictive behavior, which is determined by a host of factors including uncertain events, volatility of shocks, the individual's attitude towards risk, time preference, and addictive capital depreciation rate. They enable us to derive both qualitative and quantitative properties of the dynamics of addictive behavior and infer policy implications.

In our analysis, we explore a class of multivariate power utility functions that not only capture three basic characteristics of addictive behavior –tolerance, withdrawal and reinforcement–, but also are tractable and can thus facilitate the establishment of various properties of the model. These utility functions are called addictive multivariate power functions and admit meaningful and intuitive interpretations. For instance, in the case of harmful addiction like smoking cigarettes the total utility value from addictive consumption is always negative, whereas in the case of beneficial addiction such as music and jogging (see Stigler and Becker 1977, Becker and Murphy 1988) the utility value from addictive consumption will be positive. In the analysis of Becker and Murphy (1988) and several subsequent studies, the quadratic utility function is used as an approximation near a steady state. A major drawback of this approximation is that it is then extremely difficult to know the global pattern of dynamic addictive behavior and sometimes even locally it may not be a good approximation. Fortunately, the addictive multivariate power utility function makes the approximation obsolete and meanwhile allows us to derive closed-form solutions and obtain fresh insights into the evolution of addictive behavior and into policy issues, and to establish both global and local properties of dynamic addictive behavior.

We shall also highlight several other results. For instance, it will be easy to understand why anxiety and tension can precipitate an addiction. This is so because anxiety and tension will make the individual more present-oriented, and the more present-oriented he is, the more he will consume and thus become more addicted today rather than tomorrow. It is shown in Proposition 4 that while on the one hand, the more volatile the situation an individual faces, the more susceptible to change his addictive consumption will be, on the other hand on average his addictive consumption will decrease and it shows a similar pattern to normal goods. Our results indicate whether the individual's addictive consumption level will converge to zero, remain constant or go to infinity does not depend on his initial state but on uncertainty and his personal parameters such as his attitude to risk, time preference and addictive capital depreciation rate. Our results also offer a natural explanation of why cycles of binges and abstention attempts can occur and when cold turkey could be used to end addiction. Binges refer to a phenomenon in which an individual sporadically does too much of a particular activity, especially drinking or eating, in a short period of time. One important policy implication of our analysis is that instead of harsh treatment like going cold turkey, which can be extremely painful or sometimes even life-threatening, there always exists soft treatment–a gradual and less painful process toward addiction cessation. Another policy implication is that our theory allows a broad and flexible treatment to promote abstention: soft treatment, harsh treatment, or a combination of both.

We conclude this introduction by briefly reviewing a number of related studies in the literature. Dockner and Feichtinger (1993) study a variant of two-state-variable model of Becker and Murphy (1988) and examine cyclical consumption behavior. Orphnides and Zervos (1995) propose a discrete-time infinite horizon model of rational addiction in which individuals have unknown addictive power. They stress the informational role of experimentation and the importance of subjective beliefs and explain why addicts may regret their past consumption decisions. Recently several important behavioral models have been proposed. Laibson (2001) studies the choices of a rational decision maker who has a cue-contingent habit formation like addiction. The model shows why tastes and cravings can change rapidly from moment to moment and why the individual should actively try to avoid temptations. Gruber and Koszegi (2001) present a model of addiction that uses hyperbolic discounting preferences and exhibits forward-looking behavior. They also provide an empirical test for their model based on cigarette consumption. Bernheim and Rangel (2004) introduce a model based on three premises: addictive consumption is usually a mistake; experience with addictive substance sensitizes the involved person to environmental cues that trigger mistaken usage; addicts are well aware of their susceptibility to cue-triggered mistakes and try to respond wisely. They explore the model's positive implications for typical patterns of addiction and public policy. Compared with a large amount of empirical work on the venerable Becker-Murphy addiction model, not much empirical work has been reported on these relatively new behavioral addiction models. Our model follows most closely the model of Becker and Murphy (1988) but goes beyond their model by incorporating uncertainty in a significant and insightful way.

This paper proceeds as follows. Section 2 reviews the Becker-Murphy model and introduces preliminaries. Section 3 contains the main results including the model under uncertainty, closed-form solutions for optimal addictive consumption and capital trajectories, and their basic properties. Section 4 analyzes the deterministic case and explores policy implications. Section 5 concludes. Most of the proofs are deferred to the appendices.

2 Preliminaries

The great Chinese philosopher Confucius says: "One can gain new insight through reflecting on the past." Let us begin by briefly reviewing the Becker-Murphy model. Despite some criticisms[7], it remains a natural model that does not only capture several fundamental patterns of addictive behavior but also gives sensible predictions.

In the Becker-Murphy model, the utility of an individual at any moment t in time depends on the consumption of a normal good $z(t)$ and of an addictive good $c(t)$. The addictive good differs crucially from the normal good in that not only the current consumption of the addictive good but also its accumulated consumption $A(t)$ from the past affect the individual's utility. We can formally describe this utility function as

$$u(t) = u(c(t), A(t), z(t)), \tag{1}$$

where $c(t)$ and $z(t)$ are the current consumption of the addictive good and the normal good at time t respectively, and $A(t)$ is called *addictive capital* at time t. $A(t)$ is used as a measure to reflect the accumulated effect of the past consumption of the addictive good. The utility function u is assumed to be strictly concave of c, A, and z and to have second partial derivatives for each of the arguments.

To explain and capture three fundamental addiction characteristics: tolerance, withdrawal, and reinforcement, Becker and Murphy (1988) impose the following natural and mild conditions on the utility function u:

$$u_c > 0, \quad u_{cc} < 0. \tag{2}$$

$$u_A < 0, \quad u_{AA} < 0. \tag{3}$$

$$u_{cA} > 0. \tag{4}$$

$$u_z > 0, \quad u_{zz} < 0. \tag{5}$$

[7]See for instance Akerlof (1991, p.5). See also Winston (1980) and Schelling (1984) for different opinions about the rational approach to addictions (Stigler and Becker 1977).

Here (2) and (5) are most familiar, indicating that the marginal utilities of both addictive good and normal good are positive and decreasing. It means that the more of either good the higher utility but the lower marginal utility the individual will get. The inequality (2) describes *withdrawal* effect, implying that the individual's utility would fall should the consumption of the addictive good be reduced. The negative marginal utility of addictive capital A given by (3) captures *tolerance*, saying that less cumulative past consumption of the addictive good will enhance current utility. This assumption is markedly different from typical ones in economic theory whose marginal utilities are assumed to be positive. Like addictive good and normal good, the marginal utility of addictive capital is also decreasing. Finally, inequality (4) reflects *reinforcement* between current addictive consumption and addictive capital, stating that past consumption will bolster current consumption. Reinforcement is also called *adjacent complementarity* (see Ryder and Heal 1973, and Iannaccone 1986).

In the Becker-Murphy model, a rational addict makes a consistent plan to maximize his utility over time when he chooses his consumption bundle each time. This decision problem is formulated as

$$\begin{array}{ll} \max_{c(t),z(t)} & \int\limits_0^\infty u(c(t), A(t), z(t)) \exp(-\rho t) dt \\ \text{s.t.} & \dot{A}(t) = c(t) - \delta A(t),\ A(0) = A_0 > 0 \\ & \int\limits_0^\infty [z(t) + p_c(t)c(t)] \exp(-rt) dt \leq R_0. \end{array} \tag{6}$$

Here the objective function is an accumulation of the individual's utility over an infinite lifetime and ρ is a constant rate of time preference. The first constraint describes the addictive capital accumulation process, the parameter δ is the depreciation rate of the addictive stock over time and A_0 is the initial addictive stock. $\dot{A}(t)$ stands for the rate of change over time in A. The second constraint is the individual's budget constraint, r is the constant interest rate, R_0 is the discounted present value of lifetime income. The price of the normal good is normalized to 1, and $p_c(t)$ is the price of the addictive good at time t. The addict's goal is to select a bundle of the normal good $z(t)$ and the addictive good $c(t)$ each time t under the two constraints so as to maximize his lifetime utility.

In general, it is very difficult to obtain a closed form solution to a general optimal control problem (see Kamien and Schwartz 1991, and Sethi and Thompson 2000). Unlike other known optimal control problems, the problem (6) is especially hard to deal with due to its peculiar, unique and complex nature of the constraints (3) and (4) on the utility function u. The function u is too general to be tractable. In order to analyze the problem (6), Becker and Murphy make use of the following quadratic function as an approximation of the utility function u near a steady state

$$F(t) = \alpha_c c(t) - \alpha_A A(t) - \frac{\alpha_{cc}}{2} c^2(t) - \frac{\alpha_{AA}}{2} A^2(t) + \alpha_{cA} c(t) A(t) - \upsilon p_c c(t) \tag{7}$$

where all parameters α_c, α_A, α_{cc}, α_{AA}, α_{cA}, υ and p_c are nonnegative. We will show that using the function F greatly limits the scope of the results derived by Becker and Murphy. To see this, by using formula (16) of Becker and Murphy (1988, p.678) we have

$$\alpha_{AA} < (\rho + 2\delta)\alpha_{cA}$$

Because u is assumed to be concave, F must be concave. It follows from the Hessian matrix derived from F that $\alpha_{cc}\alpha_{AA} - \alpha_{cA}^2 \geq 0$. Then we have $\alpha_{cA}^2 \leq \alpha_{cc}\alpha_{AA}$. If we assume $\alpha_{AA} \geq \alpha_{cc}$, then $\alpha_{cA} \leq \alpha_{AA}$. To summarize the above discussion we have

Proposition 1 *In order for the function F to satisfy the concavity assumption we must have*

$$\alpha_{cA} \leq \alpha_{AA} \leq (\rho + 2\delta)\alpha_{cA}$$

provided that $\alpha_{AA} \geq \alpha_{cc}$.

This result indicates that if the depreciation rate δ and the time preference ρ are small, the range for parameters to satisfy the inequality is very narrow.[8] If $\rho + 2\delta < 1$, the quadratic function F becomes even incompatible with the basic assumption in the Becker-Murphy model. If the time preference and the depreciation rate are treated not much different from those in general consumption or investment problems, $\rho + 2\delta < 1$ seems to be a reasonable benchmark (see Weitzman 2001). This demonstrates that the quadratic function sometimes cannot be even used locally as an appropriate approximation.

3 Main Results

3.1 The Model under Uncertainty

Having the above preparations, we can now introduce the general rational addiction problem under uncertainty:

$$\begin{array}{ll} \max_{c(t),z(t)} & E\{\int_0^\infty u(A(t), c(t), z(t)) \exp(-\rho t) dt\} \\ \text{s.t.} & dA(t) = (c(t) - \delta A(t))dt + \sigma A(t) dv(t),\ A(0) = A_0 > 0 \\ & \dot{W}(t) = rW(t) - (z(t) + c(t)p_c(t)),\ W(0) = W_0 \end{array} \tag{8}$$

where all variables and parameters in (8) have the same meaning as in the basic problem (6), except for $v(t)$, σ, and $W(t)$. $W(t)$ is the wealth at time t; W_0 is the initial wealth which can be regarded as the discounted present value of a consumer's lifetime income.[9]

[8] It is also possible to give other results showing the limitation of using the quadratic function.

[9] This budget constraint is essentially the same as the one in the problem (6).

$\dot{W}(t)$ is the rate of change over time in W. The key difference from the basic model (6) is that a stochastic term $\sigma A(t)dv(t)$ is introduced here. $v(t)$ is a standard Wiener process. σ is an instantaneous volatility rate. The stochastic term is used to capture a host of random events that join forces with the intentional addictive consumption to influence addiction capital accumulation.

In the following we discuss how to determine both the optimal path of the addictive consumption and the optimal path of addictive capital. For the problem (8), the corresponding Hamilton-Jacobi-Bellman (HJB) equation is given by (see Kamien and Schwartz, 1991, p.269)

$$\rho J = \max_{c,z}\{u(A,c,z) + J_A(c-\delta A) + J_W[rW - (z + cp_c)] + \frac{1}{2}\sigma^2 A^2 J_{AA}\}, \tag{9}$$

where the parameter time t is omitted in every term and will be often ignored when no confusion can arise. The first order conditions of (9) are

$$u_c + J_A - p_c J_W = 0, \text{ and } u_z - J_W = 0. \tag{10}$$

Recall that $u_{cc} = \frac{\partial^2 u}{\partial c^2} < 0$ and $u_{zz} = \frac{\partial^2 u}{\partial z^2} < 0$. Then u_c and u_z are strictly decreasing functions with respect to c and z, respectively. So their inverse functions exist and can be written as

$$c = u_c^{-1}(p_c J_W - J_A), \text{ and } z = u_z^{-1}(J_W). \tag{11}$$

From $u_z - J_W = 0$ of (10) and (5) we know $J_W > 0$. Now we have the following simple but basic observation saying that the fundamental economics law still holds under uncertainty.

Proposition 2 *For the problem* (8) *the addictive consumption is a decreasing function of its price p_c, ceteris paribus.*

Proof: Notice that because u_c^{-1} is a strictly decreasing function, we have $\frac{\partial c}{\partial p_c}$ is negative. That is to say, the addictive consumption c will decrease if its price p_c increases. □

When a model involves uncertainty, it is useful and often necessary to focus on essential variables of the model by ignoring other nonessentials. To analyze the effect of uncertainty on addictive behavior and obtain substantial insights into the problem (8), without loss of much generality we shall assume that the normal good consumption is fixed, say, $z = 0$, and the individual has a sufficient amount of income. Now the general problem (8) becomes

$$\begin{array}{ll} \max_{c(t)} & E\{\int_0^\infty u(A(t), c(t)) \exp(-\rho t) dt\} \\ \text{s.t.} & dA(t) = (c(t) - \delta A(t))dt + \sigma A(t)dv(t),\ A(0) = A_0 > 0 \end{array} \tag{12}$$

Because this is an autonomous stochastic optimal control problem with infinite time horizon, we can assume that the value function is independent of time (see Kamien and

Schwartz 1991, p. 269). Thus by setting $J = J(A)$, we have the HJB equation as follows

$$\rho J = \max_{c}\{u(A,c) + J_A(c - \delta A) + \frac{1}{2}\sigma^2 A^2 J_{AA}\}. \tag{13}$$

The first order condition is

$$u_c + J_A = 0 \tag{14}$$

Because $\frac{\partial^2 u}{\partial c^2} < 0$ and thus u_c is a strictly decreasing function, the inverse function of u_c exists and can be given as

$$c = u_c^{-1}(-J_A). \tag{15}$$

Using (15) to substitute for c in (13) gives the HJB equation

$$\rho J = u(A, u_c^{-1}(-J_A)) + J_A(u_c^{-1}(-J_A) - \delta A) + \frac{1}{2}\sigma^2 A^2 J_{AA}. \tag{16}$$

In order to derive a closed-form formula for both the optimal path of the addictive consumption and the optimal path of the addictive capital, we shall explore the following multivariate power utility function

$$u(A,c) = -\frac{A^\beta}{c^\alpha},\ \beta \geq \alpha + 1,\ \alpha > 0. \tag{17}$$

We call this function *the addictive multivariate power utility function*, which resembles the familiar Cobb-Douglas function but has a different requirement on parameters. It is easy to calculate 1st and 2nd derivatives of this addictive utility function

$$\begin{array}{ll} u_c = \alpha\frac{A^\beta}{c^{(\alpha+1)}} > 0, & u_{cc} = -\alpha(\alpha+1)\frac{A^\beta}{c^{(\alpha+2)}} < 0 \\ u_A = -\beta\frac{A^{\beta-1}}{c^\alpha} < 0, & u_{AA} = -\beta(\beta-1)\frac{A^{\beta-2}}{c^\alpha} < 0 \\ u_{cA} = \alpha\beta\frac{A^{\beta-1}}{c^{\alpha+1}} > 0. & \end{array}$$

The following simple result shows that the addictive multivariate power utility function satisfies all the conditions (2), (3), (4), and (5) required for any addictive utility function. More importantly, this utility function makes the use of an approximation like the quadratic function F of (7) obsolete.

Proposition 3 *If $\beta \geq \alpha + 1 > 1$ and $0 < \theta < 1$, then for any given constant $\nu \geq 0$ the function $u(A, c, z) = -\frac{A^\beta}{(c+\nu)^\alpha} + z^\theta$ satisfies inequalities* (2), (3), (4), *and* (5), *and u is concave for $A > 0$, $c > 0$ and $z \geq 0$.*

3.2 Closed-Form Solutions

We shall derive an optimal solution to the problem (12) where the utility function $u(A, c)$ takes the form of (17). Now the maximization problem can be transformed into the minimization problem:

$$\begin{array}{cl} \min_{c(t)} & E\{\int_0^\infty \frac{(A(t))^\beta}{(c(t))^\alpha} \exp(-\rho t)dt\} \\ \text{s.t.} & dA(t) = (c(t) - \delta A(t))dt + \sigma A(t)dv(t),\ A(0) = A_0 > 0. \end{array}$$

Let $\tilde{J}$ be the value function of the minimization problem. It is clear that $J = -\tilde{J}$. Then the HJB equation becomes

$$\rho\tilde{J} = \min_c\{\frac{A^\beta}{c^\alpha} + \tilde{J}_A(c - \delta A) + \frac{1}{2}\sigma^2 A^2 \tilde{J}_{AA}\}. \tag{18}$$

By first order condition, we have

$$c = (\frac{\alpha A^\beta}{\tilde{J}_A})^{\frac{1}{1+\alpha}}. \tag{19}$$

Using (19) to substitute for c in (18) yields

$$\rho\tilde{J} = A^\beta((\frac{\alpha A^\beta}{\tilde{J}_A})^{\frac{1}{1+\alpha}})^{-\alpha} + \tilde{J}_A((\frac{\alpha A^\beta}{\tilde{J}_A})^{\frac{1}{1+\alpha}} - \delta A) + \frac{1}{2}\sigma^2 A^2 \tilde{J}_{AA}. \tag{20}$$

Let us guess that the value function is of the following form

$$\tilde{J}(A) = aA^\gamma,\ \gamma \neq 0.$$

Then we have

$$\tilde{J}_A = a\gamma A^{\gamma-1}, \quad \tilde{J}_{AA} = a\gamma(\gamma - 1)A^{\gamma-2}.$$

Using these formulas in (20) results in

$$\rho a A^\gamma = A^\beta((\frac{\alpha A^\beta}{a\gamma A^{\gamma-1}})^{\frac{1}{1+\alpha}})^{-\alpha} + a\gamma A^{\gamma-1}((\frac{\alpha A^\beta}{a\gamma A^{\gamma-1}})^{\frac{1}{1+\alpha}} - \delta A) + \frac{1}{2}\sigma^2 A^2 a\gamma(\gamma - 1)A^{\gamma-2}$$

which can be rewritten as

$$\rho a A^\gamma = [(\frac{a\gamma}{\alpha})^{\frac{\alpha}{1+\alpha}} + a\gamma(\frac{\alpha}{a\gamma})^{\frac{1}{1+\alpha}}]A^{\frac{\beta+\alpha\gamma-\alpha}{1+\alpha}} - \delta a\gamma A^\gamma + \frac{1}{2}\sigma^2 a\gamma(\gamma - 1)A^\gamma. \tag{21}$$

We select parameter γ to make A's power equal in every term of (21). Then we have

$$\frac{\beta + \alpha\gamma - \alpha}{1 + \alpha} = \gamma.$$

Solving this equation gives $\gamma = \beta - \alpha$. Replacing γ by $\beta - \alpha$ in (21) leads to

$$\rho a = (\frac{a(\beta-\alpha)}{\alpha})^{\frac{\alpha}{1+\alpha}} + a(\beta-\alpha)(\frac{\alpha}{a(\beta-\alpha)})^{\frac{1}{1+\alpha}} - \delta a(\beta-\alpha) + \frac{1}{2}\sigma^2 a(\beta-\alpha)(\beta-\alpha-1).$$

It follows that

$$a = (\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}}{\rho + \delta(\beta-\alpha) - \frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)})^{1+\alpha}.$$

Replacing γ by $\beta - \alpha$ and the constant a by the above formula in $\tilde{J}(A) = aA^{\gamma}$ produces an explicit value function[10]

$$\tilde{J}(A) = (\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}}{\rho + \delta(\beta-\alpha) - \frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)})^{1+\alpha} A^{\beta-\alpha}$$

Differentiating both sides of this equation with respect to A gives

$$\tilde{J}_A = (\beta-\alpha)(\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}}{\rho + \delta(\beta-\alpha) - \frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)})^{1+\alpha} A^{\beta-\alpha-1}. \tag{22}$$

$-\tilde{J}_A$ is the shadow price of addictive capital. Replacing $\tilde{J}_A$ by the above formula in (19), we can obtain the expression of the optimal addictive consumption[11]

$$c(t) = \frac{\alpha}{1+\alpha}[\frac{\rho}{\beta-\alpha} + \delta - \frac{1}{2}\sigma^2(\beta-\alpha-1)]A(t). \tag{23}$$

Using this formula to substitute for $c(t)$ in $dA(t) = (c(t) - \delta A(t))dt + \sigma A(t)dv(t)$ yields

$$dA(t) = \{\frac{\alpha}{1+\alpha}[\frac{\rho}{\beta-\alpha} - \frac{1}{2}\sigma^2(\beta-\alpha-1)] - \frac{\delta}{1+\alpha}\}A(t)dt + \sigma A(t)dv(t),\ A(0) = A_0.$$

Solving this stochastic differential equation gives

$$\begin{aligned} A(t) &= A_0 \exp\{[\tfrac{\alpha}{1+\alpha}(\tfrac{\rho}{\beta-\alpha} - \tfrac{1}{2}\sigma^2(\beta-\alpha-1)) - \tfrac{\delta}{1+\alpha} - \tfrac{\sigma^2}{2}]t + \sigma v(t)\} \\ &= A_0 \exp((\eta - \delta - \tfrac{\sigma^2}{2})t + \sigma v(t)) \end{aligned} \tag{24}$$

where $\eta = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta - \frac{1}{2}\sigma^2(\beta-\alpha-1))$. Obviously, in order for the solution $A(t)$ to be meaningful (i.e., $A(t) \geq 0$), η needs to be positive. Using $A(t)$ in (23) we obtain the optimal addictive consumption and its expectation

$$c(t) = A_0\eta \exp((\eta - \delta - \frac{\sigma^2}{2})t + \sigma v(t)) \tag{25}$$

$$E\{c(t)\} = A_0\eta \exp((\eta - \delta)t) \tag{26}$$

[10]Recall that $J(A) = -\tilde{J}(A)$. So we can obtain the analytical results about the effect of different factors on the total utility. Such results will be discussed in detail in the following sections.

[11]See the derivation of (23) in the Appendix B.

Observe that (24) and (25) have identical structures and (25) and (26) have rather similar structures. (25) gives the optimal addictive consumption trajectory which is a geometric Brownian motion, while (26) provides the expected optimal addictive consumption trajectory which is a constant with respect to t. These two formulas will be used to analyze both short-run and long-run addictive behavior. We are ready to present the following major theorem which validates the optimality of the derived solutions and whose proof is given in the Appendix C.

Theorem 1 *The formulas* (24) *and* (25) *are an optimal addictive capital trajectory and an optimal addictive consumption trajectory to the problem* (12)*, respectively.*

It is worth mentioning that although many problems arising from physics, economics and engineering etc have been naturally formulated as stochastic (control) problems, only very few closed-form solutions have been found; see Black and Scholes (1973) for their remarkable option pricing formula and Merton (1971) for a well-known explicit solution in a special case of his problem. The interested reader may refer to Fleming and Soner (2006), and Yong and Zhou (1999) on stochastic control theory in detail.

3.3 Cycles of Binges and Abstention Attempts

Binges are very common in alcohol drinking, cigarette smoking, eating and some other kinds of addiction. By binge we mean a short period of excessive indulgence in a good or an activity. Knowing the harmful effect of addiction, addicts also often attempt to reduce or quit their addictive consumption. We call this phenomenon abstention attempt. In fact cycles of binges and abstention attempts, such as overeating and dieting, are a familiar addictive behavioral pattern. Such cycles are usually irregular and are triggered by random events. Our model is capable of capturing this important feature of dynamic addictive behavior. In our model (12), random events are described by the standard Wiener process $v(t)$, $A(t)$ is a state variable, and $c(t)$ is a control variable. $v(t)$ is a random variable and directly affects the level of addictive capital $A(t)$. The individual cannot directly control his addictive capital $A(t)$ but can influence it by choosing an appropriate addictive consumption $c(t)$. In the model, beneficial events plunge the addictive capital $A(t)$ to (local) lows while harmful events compel $A(t)$ to jump to (local) highs.

Harmful random events include marital breakup, job loss, death of a loved one, and other stressful events (Becker and Murphy 1988), and environmental cues (Goldstein 1994, Laibson 2001, and Bernheim and Rangel 2004). Laibson (2001, p. 86) points out: "Such cue-based motivational effects arise in a wide range of domains, including feeding, drug use, sexual activity, social competition, aggression, and exercise/play." Certain harmful events can be happy occasions; for instance, friends gathering can create binge drinking or smoking or eating. Harmful events can induce powerful or overwhelming cravings for addictive consumption. In our model, this means that such events will instantly spur the addictive capital to reach a peak. Because of adjacent complementarity, i.e., reinforcement, between addictive capital and addictive consumption, the addict will immediately respond with a large increase of addictive consumption in order to sustain his current utility.

Beneficial events can be the death of a friend caused by addiction, a lesson of good counseling, compelling campaigns against drugs, reading of a good book on addiction control, a piece of horrific news on addiction, and etc. Such events usually appear to be sober or thought-provoking episodes. They can prompt addicts to have a strong desire to reduce or quit their addictive consumption. In our model, these events will instantly plummet the addictive capital to a bottom. Also because of reinforcement effect, addicts will immediately reciprocate with a dramatic decrease of addictive consumption in order to maintain their current utility levels.

In summary, irregular cycles of binges and abstention attempts fit well into our general framework. Furthermore, it is easy to see from the formulas (24) and (25) that the dynamic addictive consumption synchronizes with the movement of addictive capital. In fact, both consumption $c(t)$ and capital $A(t)$ movements share the same pattern with only a constant magnitude difference of η. The fluctuation of the Wiener process reflects the irregularity of cycles of binges and abstention attempts.

3.4 Properties of the Solution and the Utility Function

We will now examine the closed-form solutions given by (25) and (26) in detail and see how the parameters affect the addictive consumption and capital patterns. We will also investigate several basic properties of the general addictive multiplicative utility function.

To obtain the optimal addictive consumption path we have used the utility function $u(A,c) = -\frac{A^\beta}{c^\alpha}$. Observe that $-\alpha$ reflects the elasticity $e(c) = \frac{cu_c}{u}$ of utility over addictive consumption, and β is the elasticity $e(A) = \frac{Au_A}{u}$ of utility over addictive capital. Typically utility functions contain the same number of variables as the number of goods for consumption. It is worth stressing here that although the utility function $u(A,c)$ for the rational addiction model contains two variables A and c, it only involves one commodity c to consume. This means we need to adapt standard analysis to this context. As it is well-known, the curvature of the utility function measures the individual's attitude toward risk. For the concave utility function $u(A,c)$ the parameter $\beta - \alpha$ roughly reflects the degree of concavity and might be called *the degree of risk aversion.* The bigger $\beta - \alpha$ is, the more concave the utility function is, and the more risk averse the individual will be. Let $\psi = \beta - \alpha$. We now have the following property immediately from (26).

Proposition 4 *The expected optimal addictive consumption is a decreasing function of the level σ^2 of uncertainty ceteris paribus, i.e.,* $\frac{\partial E\{c\}}{\partial \sigma^2} < 0$.

Proposition 4 tells that the level σ^2 of uncertainty affects addictive consumption negatively on average. The more volatile the less expected consumption.

Proposition 5 *Both the optimal addictive consumption and the expected optimal addictive consumption are decreasing functions of the generalized degree ψ of risk aversion ceteris paribus, i.e.,* $\frac{\partial c}{\partial \psi} < 0$ *and* $\frac{\partial E\{c\}}{\partial \psi} < 0$.

Proposition 5 shows that as individuals become more risk averse, they incline to be less addicted to harmful substances.

Proposition 6 *The optimal addictive consumption is an increasing function of the time preference ρ, and the minus elasticity α of utility over addictive consumption, respectively, but a decreasing function of the elasticity β of utility over addictive capital ceteris paribus, i.e.,* $\frac{\partial c}{\partial \rho} > 0$, $\frac{\partial c}{\partial \alpha} > 0$, $\frac{\partial c}{\partial \beta} < 0$. *The same conclusion holds on average, i.e.,* $\frac{\partial E\{c\}}{\partial \rho} > 0$, $\frac{\partial E\{c\}}{\partial \alpha} > 0$, $\frac{\partial E\{c\}}{\partial \beta} < 0$.

In Proposition 6, the first formula $\frac{\partial c}{\partial \rho} > 0$ shows that even in the case of uncertainty present-oriented individuals tend to be more addicted to harmful goods than future-oriented individuals. This is consistent with what Becker and Murphy (1988, p. 682) observe for the case without uncertainty. The second formula $\frac{\partial c}{\partial \alpha} > 0$ indicates that individuals become more willing to consume addictive goods as elasticity $-\alpha$ is decreasing. The reason for this is that due to withdrawal effect individuals' utility would increase should the addictive consumption increase, because increasing α would reinforce withdrawal effect, it will increase their demand for addictive goods. The third formula $\frac{\partial c}{\partial \beta} < 0$ suggests that as individuals' elasticity β over addictive capital increases, they become less addicted to harmful goods. This is because marginal utility over additive capital is negative, as elasticity β increases, it will lower additive capital and as a result reduce addictive consumption because of reinforcement effect. Observe that the two parameters α and β are opposing forces for addictive consumption and are reflected in the degree of risk aversion $\beta - \alpha$.

Proposition 7 *The optimal addictive consumption is an increasing function of the depreciation rate δ for small time t but will become a deceasing function of the depreciation rate for large time t ceteris paribus, i.e.,* $\frac{\partial c(t)}{\partial \delta} = \frac{A_0}{1+\alpha}(\alpha - t\eta)\exp((\eta - \delta - \frac{\sigma^2}{2})t + \sigma v(t))$. *The same conclusion holds on average, i.e.,* $\frac{\partial E\{c(t)\}}{\partial \delta} = \frac{A_0}{1+\alpha}(\alpha - t\eta)\exp(\eta - \delta)t$.

On the one hand, it is easy to see from (25) and (26) that other things being equal a sufficiently high depreciation rate δ will drive addictive consumption to zero. On the other hand, Proposition 7 indicates that the addictive consumption increases with depreciation rate when time t is close to zero. These two conclusions appear to be contradictory. However, they are not. The reason is that $\frac{\partial c}{\partial \delta} > 0$ holds true only when time t is near zero or rather small, whereas $\frac{\partial c}{\partial \delta}$ will become negative when time t is sufficiently large. The former case is consistent with Becker and Murphy (1988, p.678) but the latter does not appear in their model. If we look at the addiction consumption pattern both in short run and in long run, it is possible that addictive consumption may first increase and then decrease with time for sufficiently large depreciation rate. An increase in δ will reduce the shadow price of addictive capital and thus increase addictive consumption; while a higher δ will also decrease the addictive capital more quickly which may reduce the tolerance effect and thus drop addictive consumption. The former force is stronger at the beginning stage, but the later force will become stronger and stronger at later stage if the depreciate rate is higher enough.

Proposition 8 *The expected addictive consumption $E\{c(t)\}$ will converge to zero as time goes to infinity in the case of $\eta - \delta < 0$, $E\{c(t)\}$ will tend to infinity with time in the case of $\eta - \delta > 0$, and $E\{c(t)\} = A_0\eta$ for all t in the case of $\eta = \delta$.*

Proposition 8 manifests two contrasting long-run addictive consumption patterns. The parameter $\eta - \delta$ could be used as *an indictor of stability*, which will be analyzed further for the deterministic case in Section 4. The case of $\eta - \delta \leq 0$ indicates that the dynamic behavior of the optimal addictive consumption is inherently stable in the sense that no matter what the individual's initial addictive capital is, his optimal addictive consumption and capital will die out in the end. The opposite case is $\eta - \delta > 0$ and shows that the individual's dynamic addictive behavior is unstable in the sense that no matter where he starts with, he will become more and more addicted and will at last totally lose control of his appetite for harmful goods. Observe that the indicator of stability $\eta - \delta = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} - \frac{1}{2}\sigma^2(\beta - \alpha - 1)) - \frac{\delta}{1+\alpha}$ is determined jointly by several basic factors α, β, δ, and σ. These factors add new elements in explaining rational addictive behavior compared with Becker and Murphy (1988). They mainly focus on the effect of initial addictive capital on the steady state of addictive capital.

When there is no uncertainty, i.e., $\sigma^2 = 0$, we have $c(t) = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)A_0 \exp[\frac{1}{1+\alpha}(\frac{\alpha\rho}{\beta-\alpha} - \delta)t]$. Then we have the following proposition for this deterministic case.

Proposition 9 *If there is no uncertainty, then $c(t) \to 0$ as $t \to \infty$ for $\frac{\alpha\rho}{\beta-\alpha} - \delta < 0$; and $c(t) \to \infty$ as $t \to \infty$ for $\frac{\alpha\rho}{\beta-\alpha} - \delta > 0$; and $c(t) = A_0\frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)(> 0)$ for all $t \geq 0$ when $\frac{\alpha\rho}{\beta-\alpha} - \delta = 0$.*

With respect to Proposition 9, the following several points are obvious. First, addictive consumption will finally converge to zero if the depreciation rate δ is sufficiently high;

If the consumer dislikes risk very much, i.e., if $\beta - \alpha$ is large enough, then addictive consumption will also finally converge to zero. However, individuals will become more addicted to harmful goods as they become more impatient. So in the case of $\frac{\alpha\rho}{\beta-\alpha} - \delta > 0$, the individual becomes sufficiently impatient and thus intends to consume more addictive good today. As a result, this will increase the addictive capital and on the other hand reinforcement effect will induce the individual to consume more in the future. Therefore as time goes to infinity, the addictive consumption can spiral out of control.

In Becker and Murphy (1998), they identify the basic conditions (2), (3), (4), and (5) that an addictive utility function should satisfy. To analyze the dynamic behavior of the addictive consumption $c(t)$ and capital $A(t)$ near a steady state, they used the quadratic function as an approximation, because a concrete addictive function was not available at that time. The use of the quadratic function, however, does not seem to be very satisfactory, because the adjacent complementary condition can be violated by the quadratic function itself, or before the state reaches the restricted ideal region. Our analysis dispenses with this approximation and is global and valid for all $c > 0$ and $A > 0$.

To derive the optimal addictive consumption we have explored the utility function of the form $u(A, c) = -\frac{A^\beta}{c^\alpha}$ with $\beta \geq \alpha + 1 > 1$, which is a special case of the multiplicative separable function $u(A, c) = f(A)g(c)$. We shall give two basic properties for this type of addictive utility function and one property for a general harmful addictive utility function.

The first result says that if the addictive utility takes the form of $u(A, c) = f(A)g(c)$ and represents a harmful addiction, then the utility value must be negative. The negative utility might be somehow surprising because many familiar utility functions typically take only positive values.

Proposition 10 *If the utility function $u(A, c) = f(A)g(c)$ satisfies conditions* (2)*,* (3) *and* (4)*, then $u(A, c) < 0$.*

Proof: If $u(A, c) = f(A)g(c) > 0$, then we must have (i) $f(A) > 0$, $g(c) > 0$ or (ii) $f(A) < 0$, $g(c) < 0$.

If (i) holds, then $u_A(A, c) = f_A(A)g(c) < 0$ by (3). Combining this with $g(c) > 0$, we have $f_A(A) < 0$. Notice further $u_{cA} = f_A(A)g_c(c) > 0$ by (4). Then we have

$$g_c(c) < 0 \tag{27}$$

On the other hand, $u_c(A, c) = f(A)g_c(c) > 0$ by (2). Combining this with $f(A) > 0$ in (i) we have $g_c(c) > 0$. Obviously, $g_c(c) > 0$ contradicts (27).

Similarly, if (ii) holds, then $f(A) < 0$. Thus we have $g_c(c) < 0$ from $u_c(A, c) = f(A)g_c(c) > 0$. Combining this with $u_{cA} = f_A(A)g_c(c) > 0$, we have

$$f_A(A) < 0 \tag{28}$$

On the other hand, we have $g(c) < 0$ from (ii). Combining this with $u_A(A,c) = f_A(A)g(c) < 0$, we have $f_A(A) > 0$ contradicts (28). In summary we have $u(A,c) = f(A)g(c) < 0$. $\square$

In contrast to harmful addiction, beneficial addiction means that the marginal utility of addiction capital is positive (Becker and Murphy 1988, p.678), i.e., $u_A > 0$ but all the other conditions in (2), (3) and (4) remain unchanged. For instance, $u(A,c) = A^\beta c^\alpha$ with $\alpha > 0$, $\beta > 0$, and $\alpha + \beta \le 1$, is a beneficial addictive function. The following result shows that the value of beneficial addictive utility is always positive.

Proposition 11 *If the utility function $u(A,c) = f(A)g(c)$ satisfies (2), (3) with $u_A > 0$, and (4), then $u(A,c) > 0$.*

The last property concerns a general harmful addictive utility function saying that the value of the utility function must be negative if the addictive capital is large enough.

Proposition 12 *Under the condition of (3), $\lim_{A\to\infty} u(c,A) = -\infty$ for any fixed addiction consumption $c = \bar{c}$.*

Proof: It follows from (3) that $\frac{\partial u(c,A)}{\partial A} < 0$ and $\frac{\partial^2 u(c,A)}{\partial A^2} < 0$ for any $c > 0$ and $A > 0$. If we let $u(\bar{c},A) = \bar{u}(A)$ for $c = \bar{c} > 0$, then $0 > \frac{\partial u(\bar{c},A)}{\partial A} = \frac{\partial \bar{u}(A)}{\partial A} = \frac{d\bar{u}(A)}{dA}$ and $0 > \frac{\partial^2 u(\bar{c},A)}{\partial A^2} = \frac{\partial}{\partial A}(\frac{\partial u(\bar{c},A)}{\partial A}) = \frac{\partial}{\partial A}(\frac{\partial \bar{u}(A)}{\partial A}) = \frac{\partial^2 \bar{u}(A)}{\partial A^2} = \frac{d^2\bar{u}(A)}{dA^2}$. Let $\frac{d\bar{u}(A)}{dA} = M(A)$. From $\frac{d^2\bar{u}(A)}{dA^2} < 0$ we have $\frac{d^2\bar{u}(A)}{dA^2} = \frac{d}{dA}\frac{d\bar{u}(A)}{dA} = \frac{dM(A)}{dA} < 0$. Let $\frac{d\bar{u}(A)}{dA}|_{A_0} = M(A_0) = M_0$ and $\bar{u}(A_0) = \bar{u}_0$ for any fixed A_0. Then we have $\frac{dM(A)}{dA} < 0$, $M(A_0) = M_0 < 0$, which implies that $M(A) < M_0$ for any $A > A_0$. Recall that $\frac{d\bar{u}(A)}{dA} = M(A)$. So we have $\frac{d\bar{u}(A)}{dA} < M_0$ for any $A > A_0$. Combining this with the initial condition $\bar{u}(A_0) = \bar{u}_0$ leads to $\bar{u}(A) < \bar{u}_0 + M_0 A$. It is easy to see that $\lim_{A\to\infty} \bar{u}(A) \le \lim_{A\to\infty} (\bar{u}_0 + M_0 A) = -\infty$. $\square$

This proposition indicates that it is impossible to construct a harmful addictive utility function which admits only positive value for all $c > 0$ and $A > 0$.

4 On Dynamics of Addiction and Policy Implications

In this section we focus on the deterministic case, derive both qualitative and quantitative properties of dynamic addictive behavior and explore policy implications. Whenever possible we also consider the effect of uncertainty. In the deterministic case, a rational addict's decision problem (12) becomes

$$\begin{array}{ll} \max_{c(t)} & \int_0^\infty -\frac{(A(t))^\beta}{(c(t))^\alpha} \exp(-\rho t)dt \\ \text{s.t.} & \dot{A}(t) = c(t) - \delta A(t), \ A(0) = A_0 > 0. \end{array} \tag{29}$$

We can use Pontryagin's maximum principle to analyze its optimal solution. The Hamiltonian function of the problem is

$$H(t, A(t), c(t), \lambda(t)) = -\frac{(A(t))^\beta}{(c(t))^\alpha} + \lambda(t)(c(t) - \delta A(t)), \tag{30}$$

where $\lambda(t)$ is the co-state variable, which represents the marginal value of the addictive capital. The ODE for $\lambda(t)$ is

$$\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial H}{\partial A} = (\rho + \delta)\lambda(t) + \frac{\beta A^{\beta-1}}{c^\alpha}. \tag{31}$$

The first order condition $\frac{\partial H}{\partial c} = 0$ implies

$$\lambda(t) = -\alpha\frac{A^\beta}{c^{\alpha+1}}. \tag{32}$$

It follows that

$$\dot{\lambda}(t) = \frac{-\alpha\beta A^{\beta-1}\dot{A}c + \alpha(\alpha+1)A^\beta \dot{c}}{c^{\alpha+2}} \tag{33}$$

In the Appendix we prove that the following transversality condition is satisfied

$$\lim_{t\to\infty} \lambda(t)A(t)e^{-\rho t} = 0 \tag{34}$$

Using $\dot{A} = c - \delta A$ and substituting (32) and (33) into (31) leads to

$$\dot{c} = \frac{c^2}{\alpha(\alpha+1)A}[\frac{\alpha\beta}{c}(c - \delta A) - (\rho + \delta)\frac{\alpha A}{c} + \beta] \tag{35}$$

As shown in the Appendix, from the above equation we can directly derive the following optimal solution of problem (29)

$$c(t) = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)A_0 \exp[\frac{1}{1+\alpha}(\frac{\alpha\rho}{\beta-\alpha} - \delta)t]. \tag{36}$$

Notice that this solution coincides with the one given by the general formula (25) for $\sigma = 0$. To verify the optimality of this solution it is sufficient to prove the concavity of the maximized Hamiltonian function with respect to A. It follows from (32) that

$$c = (\frac{\alpha A^\beta}{-\lambda})^{\frac{1}{\alpha+1}} \tag{37}$$

Substituting (37) into (30) yields the maximized Hamiltonian function

$$H^0(t, A, \lambda) = -\frac{A^{\frac{\beta}{\alpha+1}}}{(\frac{\alpha}{-\lambda})^{\frac{\alpha}{\alpha+1}}} + \lambda[(\frac{\alpha}{-\lambda})^{\frac{1}{\alpha+1}} A^{\frac{\beta}{\alpha+1}} - \delta A)] \tag{38}$$

We have the following result with respect to the maximized Hamiltonian function.

Proposition 13 *The maximized Hamiltonian function $H^0(t, A, \lambda)$ is concave in A for every t.*

4.1 General Patterns of Dynamic Addictive Behavior

We first study the qualitative properties of the dynamics of addictive behavior by the phase diagram method, which permits us to visualize how both addictive consumption and addictive capital evolve over time. For this purpose we can begin by examining the system of differential equations $\dot{c} = 0$ and $\dot{A} = 0$, which are described in detail by

$$\frac{c^2}{\alpha(\alpha+1)A}[\frac{\alpha\beta}{c}(c-\delta A) - (\rho+\delta)\frac{\alpha A}{c} + \beta] = 0 \tag{39}$$

$$c - \delta A = 0 \tag{40}$$

It follows that

$$c = \frac{\alpha}{\beta}\frac{\beta\delta+\rho+\delta}{1+\alpha}A \tag{41}$$

$$c = \delta A \tag{42}$$

In the Figures 1 and 2, addictive capital A is depicted on the horizontal axis and addictive consumption c is on the vertical axis. The lines given by equations (41) and (42) divide the first quadrant into three regions (I), (II), and (III). In each region, the arrows point in the direction of motion. For the analysis it is convenient and also sufficient to consider two distinctive cases by looking at whether $\frac{\alpha}{\beta}\frac{\beta\delta+\rho+\delta}{1+\alpha} > \delta$ or $\frac{\alpha}{\beta}\frac{\beta\delta+\rho+\delta}{1+\alpha} < \delta$. Observe that $\frac{\alpha}{\beta}\frac{\beta\delta+\rho+\delta}{1+\alpha} > \delta$ and $\frac{\alpha}{\beta}\frac{\beta\delta+\rho+\delta}{1+\alpha} < \delta$ can be written in an equivalent form, respectively, as $\frac{\alpha\rho}{\beta-\alpha} > \delta$ and $\frac{\alpha\rho}{\beta-\alpha} < \delta$. Recall that these are precisely the conditions in Proposition 9 determining whether the optimal addictive consumption trajectory is divergent or convergent.[12]

Let us first look at Figure 1, which depicts the typical case of $\frac{\alpha\rho}{\beta-\alpha} > \delta$. Because $\frac{\alpha\rho}{\beta-\alpha} > \delta$, the optimal trajectory given by the slope $c = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)A$ must lie above the line $\dot{c} = 0$, i.e., (41) which must lie above the line $\dot{A} = 0$, i.e. (42). In the figure, the parameters are given by $\alpha = 1$, $\beta = 2$, $\rho = 0.15$ and $\delta = 0.10$. In particular, the optimal trajectory $c = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)A$ can be roughly viewed as an asymptote defined by $\frac{dc}{dA} = m$ for a constant m,[13] and goes to infinity over time. It is easy to see that any trajectory originating from the area above the optimal path will go northeast and evolve into infinity. This means that if a person with an initial addictive capital $A_0 > 0$ starts with an addictive consumption no less than his optimal initial consumption c_0^*, he will be trapped in addiction and on a road to ruin. Things are not always so gloomy. The good news is that if an addict forces himself or is forced to consume the addictive good less than his optimal consumption level, he can be on a path to abstention. The figure depicts the case in which the optimal initial consumption is $c_0^* = 2.5$ when the initial capital is

[12] The case of $\frac{\alpha}{\beta}\frac{\beta\delta+\rho+\delta}{1+\alpha} = \delta$ is an easy one.

[13] The proof is given by the Appendix.

$A_0 = 20$. The optimal consumption goes to infinity as time evolves into infinity. When the initial consumption is below the optimal level, such as 2.24, 2.2246, or 1 as in the figure, the corresponding trajectory converges to the origin; however, if the initial consumption is above the optimal level, such as 2.6, the trajectory will diverge monotonically to infinity.

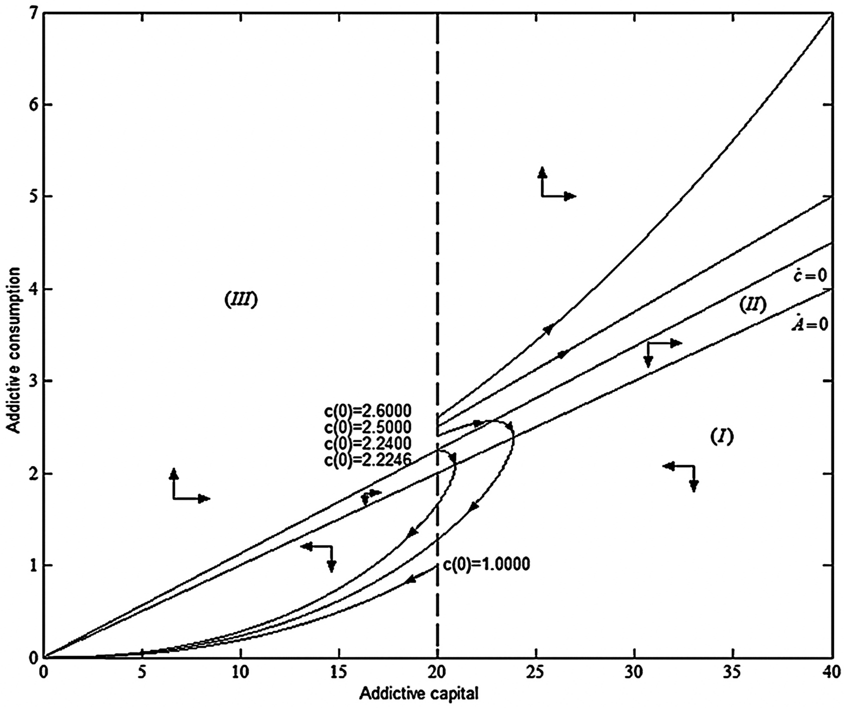

Figure 1: The case of $\frac{\alpha\rho}{\beta-\alpha} > \delta$.

Figure 2 depicts the typical case of $\frac{\alpha\rho}{\beta-\alpha} < \delta$. Because $\frac{\alpha\rho}{\beta-\alpha} < \delta$, the line $\dot{A} = 0$ must lie above the line $\dot{c} = 0$ which in turn must lie above the optimal trajectory $c = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha}+\delta)A$. In the figure, the parameters are given by $\alpha = 1$, $\beta = 2$, $\rho = 0.10$ and $\delta = 0.15$. The optimal trajectory $c = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha}+\delta)A$ converges to the origin. Again any trajectory originating from the area above the optimal trajectory will eventually go northeast and both consumption and capital levels will go to infinity. However, if an addict consumes no more than his optimal consumption level, he can totally give up his addiction. The figure depicts the case in which the optimal initial consumption is $c_0^* = 2.5$ when the initial capital is $A_0 = 20$. Other trajectories are given for consumption levels equal to 3.02, 2.75, 2.6, and 1.

Compared with Becker and Murphy (1988), our analysis has shed some new light on the evolution of additive behavior. Becker and Murphy introduce an addictive capital threshold as a factor to explain how the initial addictive capital may affect addictive consumption. According to their analysis, an addict whose initial addictive capital exceeds the threshold

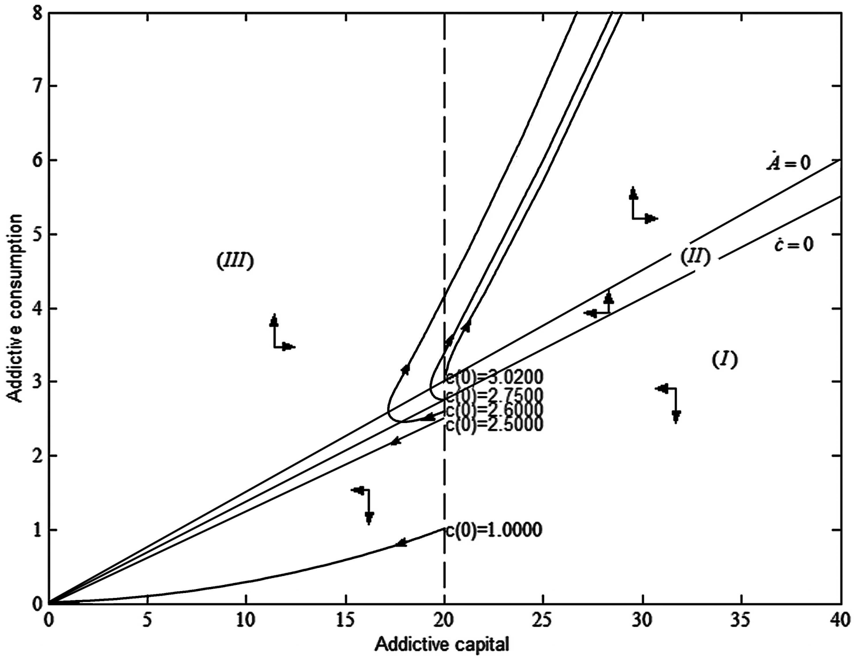

Figure 2: The case of $\frac{\alpha\rho}{\beta-\alpha} < \delta$.

will move to a higher equilibrium with higher capital and higher consumption levels,[14] but he can go to abstention if the initial capital is less than the threshold. Hence as Becker and Murphy (1988, pp. 675, 676 and 692) stress, their theory implies that strong addictions must require going "cold turkey," that is, with actions of abruptly giving up all addictive consumption. It is now known (see e.g., Hughes 2009, and Moss and Dyer 2010, ch. 6) that cold turkey is not an appropriate treatment for breaking certain addictions, because going cold turkey can cause immense withdrawal syndrome potentially resulting in death. For instance, treating alcoholics with this method can trigger life-threatening delirium tremens. In general, going cold turkey, even if not life-threatening, can be still extremely unpleasant or painful for most people, because of severe withdrawal effect. Notice that even if an addict goes cold turkey meaning that he immediately quits or dramatically reduces his addictive consumption, his addictive capital, however, cannot immediately disappear or decrease substantially. Instead it will decrease only gradually, meaning that the painful period might not be short.

Our analysis above offers fresh hope that there does exist a soft treatment-a less painful but gradual reduction process for ending addiction, no matter how long or how seriously a person has become addicted, as long as he has a will or is slightly forced to consume (even

[14]A small deviation from this equilibrium is not strong enough to jump out from the addiction trap.

just a little bit) less than his optimal addictive consumption level. Policy implications from our analysis are that in order to achieve addiction abstention, one can use soft treatment, or harsh treatment-cold turkey, or both. For instance, if a person has severe withdrawal symptoms, he can initially use soft treatment until he reaches a tolerable level from which cold turkey could be explored to terminate addictions once and for all. One caveat is that our analysis in this section has so far ignored uncertainty. In stochastic environments, even if a person currently stays in a safe zone, that is, his trajectory lies below his optimal path, random events may easily trigger him to consume more than his optimal consumption level and thus drive him into a dangerous zone. This suggests that addiction control is a complex process and requires extreme caution and constant care in order to stop addiction and prevent its relapse.

4.2 Analytical Solutions of ODE with General Initial Values

In this subsection we examine how initial addictive consumption c_0 and capital A_0 affect evolution of addictive behavior. In the optimal control problem (29), c is a control variable and A_0 is the only given initial condition. The goal was to find the optimal consumption path $c(t)$ for all $t \geq 0$. Now we ask a different but relevant question: given any initial values c_0 and A_0, how $c(t)$ and $A(t)$ will evolve over time. To provide quantitative answers, we need to solve the following system of ordinary differential equations (ODE):

$$\dot{c} = \frac{c^2}{\alpha(\alpha+1)A}[\frac{\alpha\beta}{c}(c-\delta A) - (\rho+\delta)\frac{\alpha A}{c} + \beta],\ c(0) = c_0 \tag{43}$$

$$\dot{A} = c - \delta A,\ A(0) = A_0 \tag{44}$$

Let $m_1 = \frac{1}{1+\alpha}(\beta\delta + \rho + \delta) - \delta$ and $m_2 = \frac{\beta-\alpha}{\alpha}$. Recall that $\eta = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)$. Observe that $\eta = \frac{m_1}{m_2}$. We can establish the following theorem whose proof is given in the Appendices I and J.

Theorem 2 *The solutions to* (43) *and* (44) *are*

$$A(t) = A_0[\frac{\frac{A_0}{c_0}e^{m_1 t}}{\frac{1}{\eta} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{1}{\eta})}]^{\frac{1}{m_2}} e^{-\delta t} \tag{45}$$

$$c(t) = \frac{1}{\frac{1}{\eta} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{1}{\eta})} A(t) \tag{46}$$

It is easy to verify that substituting $c_0 = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)A_0$ into (45) and (46) yields the same optimal solution as given by (36). Let c_0^* denote the optimal consumption $\frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)A_0$.

Using (45) and (46) we can derive the following two results.

Proposition 14 *In the case of* $\frac{\alpha\rho}{\beta-\alpha} > \delta$*, both the optimal addictive capital and optimal addictive consumption are increasing functions of time, that is,* $A(t)$ *and* $c(t)$ *are given by* (45) *and* (46) *with* $c_0 = \eta A_0 > 0$*. The trajectories* $(A(t), c(t))$ *given by* (45) *and* (46) *with* $c_0 > \eta A_0$ *are also increasing functions of time and will finally diverge to infinity, whereas the trajectories* $(A(t), c(t))$ *given by* (45) *and* (46) *with* $c_0 < \eta A_0$ *will be decreasing after a period of time and converge to the origin.*

Proposition 15 *In the case of* $\frac{\alpha\rho}{\beta-\alpha} < \delta$*, both the optimal addictive capital and optimal addictive consumption are decreasing functions of time and converge to the origin, that is,* $A(t)$ *and* $c(t)$ *are given by* (45) *and* (46) *with* $c_0 = \eta A_0 > 0$*. The trajectories* $(A(t), c(t))$ *given by* (45) *and* (46) *with* $c_0 < \eta A_0$ *are also decreasing functions of time and will converge to the origin, whereas the trajectories* $(A(t), c(t))$ *given by* (45) *and* (46) *with* $c_0 > \eta A_0$ *will be increasing after a period of time and finally diverge to infinity.*

From (45) and (46), we can see that both $A(t)$ and $c(t)$ are positive for all $t > 0$ if $\frac{A_0}{c_0} - \frac{1}{\eta} > 0$. Notice that $\frac{A_0}{c_0} - \frac{1}{\eta} > 0$ is equivalent to $c_0 < \eta A_0$, i.e., the initial values are inside the stable area characterized by the line of optimal solution $c = \eta A$. What is the property of trajectories originating above the line of $c = \eta A$? From Figures 1 and 2, it appears that the trajectories will diverge to infinity as $t \to \infty$. In fact, if the initial addictive consumption c_0 is given by $(\eta + \epsilon)A_0$ with $\epsilon > 0$, it is easy to calculate from (45) and (46) that both $A(t)$ and $c(t)$ can rise to any substantially high level within a relatively short period of time $t^* = -\frac{1}{m_1} \ln \frac{\varepsilon}{\eta+\varepsilon}$ (see the Appendix for its computation). This means that once an addict enters the unstable zone, his addictive consumption can easily spiral out of control. This phenomenon is consistent with the addictive behavior of many powerful substances such as opium and heroin.

We now investigate how initial values A_0 and c_0 influence a person's total utility. By substituting (45) and (46) into the objective function $\int_0^\infty -\frac{(A(t))^\beta}{(c(t))^\alpha} e^{-\rho t} dt$, we obtain the total value of utility

$$J(A_0, c_0) = -\frac{A_0^\beta}{c_0^\alpha} \frac{1+\alpha}{\rho + (\beta - \alpha)\delta}$$

Proposition 16 *For any* $0 < c_0 \le c_0^*$*, we have* $\frac{\partial J(A_0,c_0)}{\partial c_0} > 0$*,* $\frac{\partial^2 J(A_0,c_0)}{\partial c_0 \partial A_0} > 0$*, and* $\frac{\partial J(A_0,c_0)}{\partial A_0} < 0$*.*

The first inequality says that the marginal utility of initial addictive consumption c_0 is positive. This means that a utility maximizer will try to raise his initial consumption c_0 so as to increase his total utility $J(A_0, c_0)$. The second inequality states that the marginal utility of initial addictive consumption is an increasing function of initial capital A_0. That is, higher initial addictive capital A_0 will induce the agent to consume more addictive good.

As a result, it will become more difficult to give up his addiction. For example, smoking is harder to give up for those who have a long smoking history and a high addictive capital than for those who have a short history and a low capital. The last inequality means that the marginal utility of initial addictive capital A_0 is negative. That is to say, the person would be quite happy to reduce his initial capital, should it be possible for him to do so. We should, however, keep in mind that an addict cannot directly control his addictive capital being a state variable, and he can only influence his addictive capital by managing his addictive consumption being a control variable. Because of the second relation, lowering capital A_0 will also lower consumption c_0. The three inequalities give a mixed message about addiction control. A final point is that even with the same initial addictive A_0, the difficulty of addiction control varies from person to person, because each person's utility depends on his personal factors α, β, ρ, and δ.

5 Conclusion

Addictions are fundamental and perplexing types of human behavior, and are extremely difficult to model because of their unique and mysterious nature. In a path-breaking paper, Becker and Murphy (1988) develop a general rational choice framework for studying the dynamic behavior of addictive consumption. They model addictions in a deterministic environment as habit formation by capturing three fundamental features of addictive behavior: tolerance, withdrawal and reinforcement. The theoretical and empirical insights provided by Becker and Murphy (1988) and many subsequent studies have considerably advanced our understanding of addiction and also influenced decision makers on addiction control policy.

The current paper has developed a more realistic rational addiction framework that not only maintains the essential features of the Becker-Murphy model: tolerance, withdrawal and reinforcement, but also takes a crucial factor –uncertainty– that addicts have to face more often than nonaddicts do, into consideration. It is widely observed that random events such as divorce, unemployment, death of a loved one, peer pressure, exposure to environmental cues, etc, can precipitate and exacerbate an addiction, while sober and thought-provoking events such as the death of a friend caused by addiction and compelling campaigns against drugs, can discourage addiction. In our new framework uncertainty is modeled through the Wiener stochastic process. Somewhat surprisingly we have been able to derive closed-form expressions for both the optimal and expected optimal addictive consumption and capital trajectories. These solutions enable us to derive both global and local, qualitative and quantitative, properties of dynamic addictive behavior.

Our results explain several typical patterns of addictive behavior including cycles of binges and abstention attempts, and also have novel implications for addiction control policy. For instance, in contrast to the theory of Becker and Murphy implying that cold turkey must be used to terminate severe addictions, our theory offers an alternative approach-a soft treatment. That is, we demonstrate that there always exists a gradual and less painful process for ending addiction. We believe this insight is both important and practical for policy implementation, because harsh treatment-like going cold turkey can be extremely dangerous or even life-threatening. To achieve addiction abstention, our theory permits a flexible and comprehensive approach: the use of soft treatment, or harsh treatment, or a combination of both.

In our analysis, we have made good use of the addictive multivariate power utility functions that not only capture three basic characteristics of addictive behavior: tolerance, withdrawal and reinforcement, but also admit intuitive interpretations and facilitate the analysis. Becker and Murphy (1988) first propose general conditions that meet the three basic characteristics of addictive behavior and then use a quadratic utility function as an approximation to carry out their analysis. In this way it is very difficult if not impossible to derive global properties of dynamic addictive behavior. In fact, a number of subsequent studies have also explored similar quadratic functions as an approximation. The great advantage of the addictive multivariate power utility function is that it dispenses with the approximation, admits natural and meaningful interpretations, and more importantly it helps to achieve explicit solutions and therefore permits us to gain clear insights. Even in the deterministic case, we also obtain substantial new results that go beyond the existing ones.

The current study also leaves us with some open questions. For the purpose of treatment and prevention of addiction, it would be a great advance if scientists could invent a device or method to detect and gauge the level of addictive capital in the human body. Recall that reinforcement, tolerance and withdrawal are part of the widely accepted and observed syndrome of addiction. The rational models of Becker and Murphy and subsequent researchers have well captured these fundamental features. It is our conjecture that the 'substance' of the addictive capital should exist in an addictive human body. As pointed out previously, because of the incorporation of uncertainty, we have ignored the effect of a normal good in our analysis. It is, however, reasonable to expect that many insights obtained in this article could carry over to the more general model with a normal good. We believe that the analysis developed here for the basic model provides a useful and necessary foundation for the study of the more general problem. How to solve the general problem remains a challenge. Resolving this problem will yield a better understanding of how pricing and/or taxing on the addictive good affects the addictive behavior.

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Appendix

A Proof of Proposition 3

Since $u(A,c,z) = -\frac{A^\beta}{(c+\nu)^\alpha} + z^\theta$, it is sufficient to prove that $u(A,c) = -\frac{A^\beta}{(c+\nu)^\alpha}$ is concave in A and c. Then we have

$$u_c = \alpha\frac{A^\beta}{(c+\nu)^{(\alpha+1)}} > 0, \quad u_{cc} = -\alpha(\alpha+1)\frac{A^\beta}{(c+\nu)^{(\alpha+2)}} < 0, \tag{47}$$

$$u_A = -\beta\frac{A^{\beta-1}}{(c+\nu)^\alpha} < 0, \quad u_{AA} = -\beta(\beta-1)\frac{A^{\beta-2}}{(c+\nu)^\alpha} < 0, \tag{48}$$

$$u_{cA} = \alpha\beta\frac{A^{\beta-1}}{(c+\nu)^{(\alpha+1)}} > 0 \tag{49}$$

Then, from the Hessian matrix

$$H = \begin{pmatrix} u_{AA} & u_{Ac} \\ u_{cA} & u_{cc} \end{pmatrix} = \begin{pmatrix} -\beta(\beta-1)\frac{A^{\beta-2}}{(c+\nu)^\alpha} & \alpha\beta\frac{A^{\beta-1}}{(c+\nu)^{(\alpha+1)}} \\ \alpha\beta\frac{A^{\beta-1}}{(c+\nu)^{(\alpha+1)}} & -\alpha(\alpha+1)\frac{A^\beta}{(c+\nu)^{(\alpha+2)}} \end{pmatrix} \tag{50}$$

We have $u_{AA} < 0$, $u_{cc} < 0$ and

$$\begin{aligned} u_{cc}u_{AA} - (u_{cA})^2 &= [-\alpha(\alpha+1)\tfrac{A^\beta}{(c+\nu)^{(\alpha+2)}}][-\beta(\beta-1)\tfrac{A^{\beta-2}}{(c+\nu)^\alpha}] - [\alpha\beta\tfrac{A^{\beta-1}}{(c+\nu)^{-(\alpha+1)}}]^2 \\ &= \alpha(\alpha+1)\beta(\beta-1)\tfrac{A^{2\beta-2}}{(c+\nu)^{(2\alpha+2)}} - \alpha^2\beta^2\tfrac{A^{2\beta-2}}{(c+\nu)^{(2\alpha+2)}} \\ &= [(\alpha+1)(\beta-1)\alpha\beta - \alpha^2\beta^2]\tfrac{A^{2\beta-2}}{(c+\nu)^{(2\alpha+2)}} \\ &= [(\alpha+1)(\beta-1) - \alpha\beta]\alpha\beta\tfrac{A^{2\beta-2}}{(c+\nu)^{(2\alpha+2)}} \\ &= [\alpha\beta - \alpha + \beta - 1 - \alpha\beta]\alpha\beta\tfrac{A^{2\beta-2}}{(c+\nu)^{(2\alpha+2)}} \\ &= [\beta - (\alpha+1)]\alpha\beta\tfrac{A^{2\beta-2}}{(c+\nu)^{(2\alpha+2)}} \geq 0 \end{aligned} \tag{51}$$

So H is negative semi-definite under the condition of $\beta - (\alpha+1) \geq 0$. In other words, the utility function $-\frac{A^\beta}{(c+\nu)^\alpha}$ is concave for $\beta \geq \alpha + 1 > 1$. □

B The Derivation of the Formula (23)

Substituting (22) into (19), we have

$$c(t) = \{\alpha A(t)^\beta/[(\beta-\alpha)(\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}}{\rho+\delta(\beta-\alpha)-\frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)})^{1+\alpha}A(t)^{\beta-\alpha-1}]\}^{\frac{1}{1+\alpha}}$$

$$= \alpha^{\frac{1}{1+\alpha}}A(t)/[(\beta-\alpha)^{\frac{1}{1+\alpha}}\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}}{\rho+\delta(\beta-\alpha)-\frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)}]$$

$$= \frac{(\frac{\alpha}{\beta-\alpha})^{\frac{1}{1+\alpha}}[\rho+\delta(\beta-\alpha)-\frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)]}{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}} + (\beta-\alpha)^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}}A(t)$$

$$= \frac{\alpha[\rho+\delta(\beta-\alpha)-\frac{1}{2}\sigma^2(\beta-\alpha)(\beta-\alpha-1)]}{(\beta-\alpha)+(\beta-\alpha)\alpha}A(t)$$

$$= \frac{\alpha}{1+\alpha}[\frac{\rho}{\beta-\alpha}+\delta-\frac{1}{2}\sigma^2(\beta-\alpha-1)]A(t)$$

□

C Proof of Theorem 1

For $A > 0$ and $T > 0$, let $(A(t), c(t))$ be an arbitrary admissible addictive capital and consumption. Let $V(\cdot)$ denote an arbitrary solution of the HJB equation (18). Applying the well-known Ito Lemma to the function $w(t, A) = e^{-\rho t}V(A)$ yields

$$\begin{aligned}
dw &= w_t dt + w_A dA + \tfrac{1}{2}w_{AA}(dA)^2 \\
&= -\rho e^{-\rho t}V(A)dt + e^{-\rho t}(V_A dA + \tfrac{1}{2}V_{AA}dA\cdot dA) \\
&= -\rho e^{-\rho t}V(A)dt + e^{-\rho t}\{V_A[(c(t)-\delta A(t))dt + \delta A(t)dv(t)] \\
&\quad +\tfrac{1}{2}V_{AA}[(c(t)-\delta A(t))dt + \sigma A(t)dv(t)]^2\} \\
&= -\rho e^{-\rho t}V(A)dt + e^{-\rho t}\{V_A[(c(t)-\delta A(t))dt + \delta A(t)dv(t)] + \tfrac{1}{2}V_{AA}\sigma^2(A(t))^2dt\}
\end{aligned}$$

where we use the rules $(dt)^2 = dt\cdot dv(t) = dv(t)\cdot dt = 0$ and $dv(t)\cdot dv(t) = dt$. Then we have

$$E\{w(T,A(T)) - w(0,A(0))\} = E\{\int_0^T e^{-\rho t}[-\rho V(A(t)) + V_A(c(t)-\delta A(t)) + \tfrac{1}{2}\sigma^2(A(t))^2V_{AA}]dt\}$$

$$= E\{\int_0^T e^{-\rho t}[-\rho V(A(t)) + V_A(c(t)-\delta A(t)) + \tfrac{1}{2}\sigma^2(A(t))^2V_{AA} + \tfrac{(A(t))^\beta}{(c(t))^\alpha} - \tfrac{(A(t))^\beta}{(c(t))^\alpha}]dt\}$$

$$= E\{\int_0^T e^{-\rho t}[-\rho V(A(t)) + V_A(c(t)-\delta A(t)) + \tfrac{1}{2}\sigma^2(A(t))^2V_{AA} + \tfrac{(A(t))^\beta}{(c(t))^\alpha}]dt\} - E\{\int_0^T e^{-\rho t}\tfrac{(A(t))^\beta}{(c(t))^\alpha}dt\}$$

where we use the formula $E\{\int_0^T \delta A(t)dv(t)\} = 0$. Furthermore, we obtain

$$\begin{array}{rcl} w(0, A(0)) & = & E\{w(T, A(T))\} \\ & & -E\{\int\limits_0^T e^{-\rho t}[-\rho V(A(t)) + V_A(c(t) - \delta A(t)) + \frac{1}{2}\sigma^2(A(t))^2 V_{AA} + \frac{(A(t))^\beta}{(c(t))^\alpha}]dt\} \\ & & +E\{\int\limits_0^T e^{-\rho t}\frac{(A(t))^\beta}{(c(t))^\alpha}dt\} \end{array}$$

From the HJB equation (18) in Section 3.2, we know that

$$-\rho V(A(t)) + V_A(c(t) - \delta A(t)) + \frac{1}{2}\sigma^2(A(t))^2 V_{AA} + \frac{(A(t))^\beta}{(c(t))^\alpha} \geq 0.$$

Then

$$V(A_0) = w(0, A(0)) \leq E\{w(T, A(T))\} + E\{\int\limits_0^T e^{-\rho t}\frac{(A(t))^\beta}{(c(t))^\alpha}dt\} \tag{52}$$

By using the transversality condition

$$\lim_{T\to\infty} E\{w(T, A(T))\} = 0$$

and

$$\lim_{T\to\infty} E\{\int\limits_0^T e^{-\rho t}\frac{(A(t))^\beta}{(c(t))^\alpha}dt\} = E\{\int\limits_0^\infty e^{-\rho t}\frac{(A(t))^\beta}{(c(t))^\alpha}dt\}$$

we can derive from the inequality (52) that

$$V(A_0) \leq E\{\int\limits_0^\infty e^{-\rho t}\frac{(A(t))^\beta}{(c(t))^\alpha}dt\} \tag{53}$$

Since (53) holds for any admissible addictive capital $A(t)$ and consumption $c(t)$, we have

$$V(A_0) \leq \min_{c(t)} E\{\int\limits_0^\infty e^{-\rho t}\frac{(A(t))^\beta}{(c(t))^\alpha}dt\} = \tilde{J}$$

On the other hand, let $A^*(t)$ and $c^*(t)$ be given respectively by (24) and (25) in Section 3.2. Recall that to derive $A^*(t)$ and $c^*(t)$, we have used the form of $V(A) = aA^\gamma$. Observe that

$$-\rho V(A^*(t)) + V_A(c^*(t) - \delta A^*(t)) + \frac{1}{2}\sigma^2(A^*(t))^2 V_{AA} + \frac{(A^*(t))^\beta}{(c^*(t))^\alpha} = 0$$

This means that the HJB equation is satisfied by $(A(t), c(t))$. Therefore,

$$\begin{aligned} V(A_0) &= w(0, A(0)) \\ &= E\{w(T, A^*(T))\} + E\{\int_0^T e^{-\rho t} \tfrac{(A^*(t))^\beta}{(c^*(t))^\alpha} dt\} \\ &= E\{e^{-\rho T} V(A^*(T))\} + E\{\int_0^T e^{-\rho t} \tfrac{(A^*(t))^\beta}{(c^*(t))^\alpha} dt\} \end{aligned}$$

It remains to show that $\lim_{T\to\infty} E\{w(T, A^*(T))\} = 0$. Notice that

$$\begin{aligned} \lim_{T\to\infty} E\{w(T, A^*(T))\} &= \lim_{T\to\infty} E\{e^{-\rho T} a A^*(T)^{\beta-\alpha}\} \\ &= \lim_{T\to\infty} aE\{\exp(-\rho T) A_0^{\beta-\alpha} \exp\{[\tfrac{\alpha}{1+\alpha}(\tfrac{\rho}{\beta-\alpha} - \tfrac{1}{2}\sigma^2(\beta-\alpha-1)) \\ &\quad -\tfrac{\delta}{1+\alpha} - \tfrac{\sigma^2}{2}]T + \sigma v(T)\}^{\beta-\alpha}\} \\ &= \lim_{T\to\infty} aA_0^{\beta-\alpha} E\{\exp\{-\rho T + (\beta-\alpha)[\tfrac{\alpha}{1+\alpha}(\tfrac{\rho}{\beta-\alpha} \\ &\quad -\tfrac{1}{2}\sigma^2(\beta-\alpha-1)) - \tfrac{\delta}{1+\alpha} - \tfrac{\sigma^2}{2}]T + \sigma(\beta-\alpha)v(T)\}\} \\ &= \lim_{T\to\infty} aA_0^{\beta-\alpha} \exp\{-(\beta-\alpha)[\tfrac{\rho+\delta}{1+\alpha} + \tfrac{\alpha}{2(1+\alpha)}\sigma^2(\beta-\alpha-1)]T\} \\ &= 0 \end{aligned}$$

Then we have

$$\begin{aligned} V(A_0) &= \lim_{T\to\infty} E\{\int_0^T e^{-\rho t} \tfrac{(A^*(t))^\beta}{(c^*(t))^\alpha} dt\} \\ &= E\{\int_0^\infty e^{-\rho t} \tfrac{(A^*(t))^\beta}{(c^*(t))^\alpha} dt\} \\ &\geq \min E\{\int_0^\infty e^{-\rho t} \tfrac{(A(t))^\beta}{(c(t))^\alpha} dt\} \\ &= \tilde{J} \end{aligned} \tag{54}$$

By (53) and (54), $V(A_0) = \tilde{J}$. □

D Proof of Proposition 5

We prove $\frac{\partial c}{\partial \psi} < 0$. The same argument works for $\frac{\partial E\{c\}}{\partial \psi} < 0$. Plugging $\psi = \beta - \alpha$ into $c(t)$ of (25) yields

$$c(t) = A_0(\frac{\beta-\psi}{1+\beta-\psi})(\frac{\rho}{\psi} + \delta - \frac{1}{2}\sigma^2(\psi-1))e^{\{[(\frac{\beta-\psi}{1+\beta-\psi})(\frac{\rho}{\psi}+\delta-\frac{1}{2}\sigma^2(\psi-1))-\delta-\frac{\sigma^2}{2}]t+\sigma v(t)\}}.$$

Then we have

$$\begin{aligned} \tfrac{\partial c}{\partial \psi} &= A_0[\tfrac{-1}{(1+\beta-\psi)^2}(\tfrac{\rho}{\psi} + \delta - \tfrac{\sigma^2}{2}(\psi-1)) + \tfrac{\beta-\psi}{1+\beta-\psi}(-\tfrac{\rho}{\psi^2} - \tfrac{\sigma^2}{2})](1+\eta t)e^{[(\eta-\delta-\frac{\sigma^2}{2})t+\sigma v(t)]} \\ &= -A_0[\tfrac{\eta}{\alpha(1+\alpha)} + \tfrac{\alpha}{(1+\alpha)^2}(\tfrac{\rho}{(\beta-\alpha)^2} + \tfrac{\sigma^2}{2})](1+\eta t)e^{[(\eta-\delta-\frac{\sigma^2}{2})t+\sigma v(t)]} \\ &< 0 \end{aligned}$$

as long as $\alpha > 0$, $\eta > 0$ and $t \geq 0$. □

E The Optimal Solution of the Case of $\sigma = 0$

Consider the system of equations:

$$\dot{c} = \frac{c}{\alpha(1+\alpha)A}[\alpha\beta(c - \delta A) - \alpha(\rho + \delta)A + \beta c] \tag{55}$$

$$\dot{A} = c - \delta A, \quad A(0) = A_0 \tag{56}$$

We guess the optimal addictive consumption is a linear function of addictive capital, i.e. $c(t) = \tilde{\eta}A(t)$, where $\tilde{\eta}$ is a positive constant to be determined. Substituting it into (56) leads to

$$\dot{A} = \tilde{\eta}A - \delta A = (\tilde{\eta} - \delta)A, \; A(0) = A_0 \tag{57}$$

Solving (57) for $A(t)$ yields

$$A(t) = A_0 e^{(\tilde{\eta}-\delta)t} \tag{58}$$

Then

$$c(t) = \tilde{\eta}A(t) = \tilde{\eta}A_0 e^{(\tilde{\eta}-\delta)t} \tag{59}$$

Substituting (58) and (59) into RHS of (55) leads to

$$\begin{aligned} &\tfrac{c}{\alpha(1+\alpha)A}[\alpha\beta(c - \delta A) - \alpha(\rho+\delta)A + \beta c] \\ &= \tfrac{\tilde{\eta}A_0 e^{(\tilde{\eta}-\delta)t}}{\alpha(1+\alpha)A_0 e^{(\tilde{\eta}-\delta)t}}[\alpha\beta(\tilde{\eta}A_0 e^{(\tilde{\eta}-\delta)t} - \delta A_0 e^{(\tilde{\eta}-\delta)t}) - \alpha(\rho+\delta)A_0 e^{(\tilde{\eta}-\delta)t} + \beta\tilde{\eta}A_0 e^{(\tilde{\eta}-\delta)t}] \\ &= \tfrac{\tilde{\eta}}{\alpha(1+\alpha)}[\alpha\beta(\tilde{\eta}-\delta) - \alpha(\rho+\delta) + \beta\tilde{\eta}]A_0 e^{(\tilde{\eta}-\delta)t} \end{aligned} \tag{60}$$

Recall that $\dot{c}(t) = \tilde{\eta}(\tilde{\eta} - \delta)A_0 e^{(\tilde{\eta}-\delta)t}$ by (58). Similarly, substituting it into LHS of (55) gives

$$\dot{c}(t) = \tilde{\eta}(\tilde{\eta} - \delta)A_0 e^{(\tilde{\eta}-\delta)t} \tag{61}$$

With the help of (55), (60), and (61), we have

$$\tilde{\eta}(\tilde{\eta} - \delta)A_0 e^{(\tilde{\eta}-\delta)t} = \frac{\tilde{\eta}}{\alpha(1+\alpha)}[\alpha\beta(\tilde{\eta}-\delta) - \alpha(\rho+\delta) + \beta\tilde{\eta}]A_0 e^{(\tilde{\eta}-\delta)t} \tag{62}$$

From (62) we have

$$\begin{aligned}\tilde{\eta} &= \tfrac{1}{\alpha(1+\alpha)}[\alpha\beta(\tilde{\eta}-\alpha\beta)-\alpha(\rho+\delta)+\beta\tilde{\eta}]+\delta \\ &= \tfrac{\alpha\beta+\beta}{\alpha(1+\alpha)}\tilde{\eta}-\tfrac{1}{\alpha(1+\alpha)}[\alpha\beta\delta+\alpha(\rho+\delta)]+\delta \\ &= \tfrac{\beta}{\alpha}\tilde{\eta}-\tfrac{1}{1+\alpha}(\beta\delta+\rho+\delta)+\delta\end{aligned} \tag{63}$$

Then

$$(\frac{\beta}{\alpha}-1)\tilde{\eta} = \frac{1}{1+\alpha}(\beta\delta+\rho+\delta)-\delta \tag{64}$$

Solving for $\tilde{\eta}$ in (64), we have

$$\begin{aligned}\tilde{\eta} &= \tfrac{\alpha}{\beta-\alpha}[\tfrac{1}{1+\alpha}(\beta\delta+\rho+\delta)-\delta] = \tfrac{\alpha}{\beta-\alpha}(\tfrac{\beta\delta}{1+\alpha}+\tfrac{\rho}{1+\alpha}+\tfrac{\delta}{1+\alpha}-\delta) \\ &= \tfrac{\alpha}{\beta-\alpha}(\tfrac{\beta\delta}{1+\alpha}+\tfrac{\rho}{1+\alpha}-\tfrac{\alpha\delta}{1+\alpha}) \\ &= \tfrac{\alpha}{\beta-\alpha}[\tfrac{(\beta-\alpha)\delta}{1+\alpha}+\tfrac{\rho}{1+\alpha}] \\ &= \tfrac{\alpha}{1+\alpha}(\tfrac{\rho}{\beta-\alpha}+\delta)\end{aligned} \tag{65}$$

Notice that $\tilde{\eta}$ is identical to η if we set $\sigma=0$ in the expression of η. In the meantime, it is not hard to check the transversality condition $\lim_{t\to\infty}\lambda(t)A(t)e^{-\rho t}=0$; see the following Appendix.

By Proposition 13, the solution is a global optimal solution. □

F Proof of the Transversality Condition (34)

Recall from (32) that

$$\lambda(t) = -\alpha\frac{A(t)^{\beta}}{c(t)^{\alpha+1}} \tag{66}$$

Substituting (66) into $\lim_{t\to\infty}\lambda(t)A(t)e^{-\rho t}$ leads to

$$\lim_{t\to\infty}\lambda(t)A(t)e^{-\rho t} = -\lim_{t\to\infty}\alpha\frac{A(t)^{\beta}}{c(t)^{\alpha+1}}A(t)e^{-\rho t} \tag{67}$$

Substituting (58) and (59) into (67), we obtain

$$\lim_{t\to\infty}\alpha\frac{A(t)^{\beta}}{c(t)^{\alpha+1}}A(t)e^{-\rho t} = \lim_{t\to\infty}\alpha\tilde{\eta}^{\alpha+1}[A_0e^{(\tilde{\eta}-\delta)t}]^{\beta-\alpha}e^{-\rho t} \tag{68}$$

Then

$$
\begin{aligned}
&\lim_{t\to\infty} \alpha\tilde{\eta}^{\alpha+1}[A_0 e^{(\tilde{\eta}-\delta)t}]^{\beta-\alpha} e^{-\rho t} = \lim_{t\to\infty} \alpha\tilde{\eta}^{\alpha+1} A_0^{\beta-\alpha} e^{(\beta-\alpha)(\tilde{\eta}-\delta)t} e^{-\rho t} \\
&= \lim_{t\to\infty} \alpha[\tfrac{\alpha}{1+\alpha}(\tfrac{\rho}{\beta-\alpha}+\delta)]^{\alpha+1} A_0^{\beta-\alpha} e^{(\beta-\alpha)[\frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha}+\delta)-\delta]t} e^{-\rho t} \\
&= \lim_{t\to\infty} \alpha[\tfrac{\alpha}{1+\alpha}(\tfrac{\rho}{\beta-\alpha}+\delta)]^{\alpha+1} A_0^{\beta-\alpha} e^{\{\frac{\alpha}{1+\alpha}[\rho+(\beta-\alpha)\delta]-(\beta-\alpha)\delta\}t} e^{-\rho t} \\
&= \lim_{t\to\infty} \alpha[\tfrac{\alpha}{1+\alpha}(\tfrac{\rho}{\beta-\alpha}+\delta)]^{\alpha+1} A_0^{\beta-\alpha} e^{-\frac{1}{1+\alpha}[\rho+(\beta-\alpha)\delta]t} = 0
\end{aligned}
\tag{69}
$$

With the help of (67)-(69), we have $\lim_{t\to\infty} \lambda(t)A(t)e^{-\rho t} = 0$. □

We also provide a second method of proving the above result. The method will be referred in a later result.

From (67), we have

$$
\lim_{t\to\infty} \alpha \frac{A(t)^\beta}{c(t)^{\alpha+1}} A(t) e^{-\rho t} = \alpha \lim_{t\to\infty} \frac{A(t)^\beta}{c(t)^\alpha} \frac{A(t)}{c(t)} e^{-\rho t} \tag{70}
$$

Combing this with (103) leads to

$$
\lim_{t\to\infty} \frac{A(t)^\beta}{c(t)^\alpha} \frac{A(t)}{c(t)} e^{-\rho t} = \lim_{t\to\infty} A_0^{\beta-\alpha} (\frac{A_0}{c_0})^\alpha e^{-\frac{1}{1+\alpha}[\rho+(\beta-\alpha)\delta]t} \frac{A(t)}{c(t)} dt \tag{71}
$$

Substituting (97) into (71), we have

$$
\begin{aligned}
&\lim_{t\to\infty} A_0^{\beta-\alpha} (\tfrac{A_0}{c_0})^\alpha e^{-\frac{1}{1+\alpha}[\rho+(\beta-\alpha)\delta]t} \tfrac{A(t)}{c(t)} \\
&= \lim_{t\to\infty} A_0^{\beta-\alpha} (\tfrac{A_0}{c_0})^\alpha e^{-\frac{1}{1+\alpha}[\rho+(\beta-\alpha)\delta]t} [\tfrac{1}{\eta} + e^{m_1 t}(\tfrac{A_0}{c_0} - \tfrac{1}{\eta})] \\
&= \lim_{t\to\infty} A_0^{\beta-\alpha} (\tfrac{A_0}{c_0})^\alpha e^{-m_1 t} [\tfrac{1}{\eta} + e^{m_1 t}(\tfrac{A_0}{c_0} - \tfrac{1}{\eta})] \\
&= \lim_{t\to\infty} A_0^{\beta-\alpha} (\tfrac{A_0}{c_0})^\alpha (\tfrac{1}{\eta} e^{-m_1 t} + \tfrac{A_0}{c_0} - \tfrac{1}{\eta})
\end{aligned}
\tag{72}
$$

Obviously, c_0 must be equal to ηA_0, i.e., $\frac{A_0}{c_0} = \frac{1}{\eta}$, in order for the RHS of (72) to be zero. □

G Concavity of $H^0(t, A, \lambda)$

The second order derivative of the maximized Hamiltonian function $H^0(t, \lambda, A)$ of (38) is given by

$$
\frac{\partial H_0^2}{\partial A^2} = -\frac{\frac{\beta}{\alpha+1}(\frac{\beta}{\alpha+1}-1)A^{\frac{\beta}{\alpha+1}-2}}{(\frac{\alpha}{-\lambda})^{\frac{\alpha}{\alpha+1}}} + \lambda[(\frac{\alpha}{-\lambda})^{\frac{1}{\alpha+1}} \frac{\beta}{\alpha+1}(\frac{\beta}{\alpha+1}-1)A^{\frac{\beta}{\alpha+1}-2}] \tag{73}
$$

Notice that (32) implies $\lambda(t)$ is negative for each t. Combining this with (73), we know $\frac{\partial H_0^2}{\partial A^2} \leq 0$ if $\beta \geq \alpha + 1 > 1$. Then the maximized Hamilton function $H^0(t, \lambda, A)$ is concave in A for each t under the condition of $\beta \geq \alpha + 1$. □

H Asymptote

From (55) and (56), we have

$$\frac{dc}{dA} = \frac{\frac{c}{\alpha(1+\alpha)A}[\alpha\beta(c - \delta A) - \alpha(\rho + \delta)A + \beta c]}{c - \delta A} \tag{74}$$

Assume $c = mA$. Then $\frac{dc}{dA} = m$, and

$$\frac{\frac{mA}{\alpha(1+\alpha)A}[\alpha\beta(mA - \delta A) - \alpha(\rho + \delta)A + \beta mA]}{mA - \delta A} = m \tag{75}$$

Solving for m in (75) yields

$$m = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta - \alpha} + \delta) \tag{76}$$

Then the asymptote can be expressed as

$$c(t) = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta - \alpha} + \delta)A(t) \tag{77}$$

which is exactly the optimal solution of the addictive problem. □

I Part 1 of the Proof of Theorem 2

Consider the following ODEs (78) and (79):

$$\dot{c} = \frac{c^2}{\alpha(1+\alpha)A}[\frac{\alpha\beta}{c}(c - \delta A) - (\rho + \delta)\frac{\alpha A}{c} + \beta], \quad c(0) = c_0 \tag{78}$$

$$\dot{A} = c - \delta A, \quad A(0) = A_0 \tag{79}$$

Here, we should keep in mind that $c(0) = c_0$ is a general initial value of addictive consumption, not just the optimal one. From (78) and (79), we have

$$
\begin{aligned}
\frac{\dot{c}}{c} &= \tfrac{1}{\alpha(1+\alpha)}\tfrac{c}{A}[\alpha\beta(1-\delta\tfrac{A}{c}) - \alpha(\rho+\delta)\tfrac{A}{c} + \beta] \\
&= \tfrac{1}{\alpha(1+\alpha)}\tfrac{c}{A}\{\alpha\beta + \beta - [\alpha\beta\delta + \alpha(\rho+\delta)]\tfrac{A}{c}\} \\
&= \tfrac{1}{\alpha(1+\alpha)}\{(\alpha\beta+\beta)\tfrac{c}{A} - [\alpha\beta\delta + \alpha(\rho+\delta)]\} \\
&= \{\tfrac{1}{\alpha(1+\alpha)}(\alpha\beta+\beta)\tfrac{c}{A} - \tfrac{1}{\alpha(1+\alpha)}[\alpha\beta\delta + \alpha(\rho+\delta)]\} \\
&= [\tfrac{\beta}{\alpha}\tfrac{c}{A} - \tfrac{1}{1+\alpha}(\beta\delta + \rho + \delta)]
\end{aligned}
\tag{80}
$$

and

$$
\frac{\dot{A}}{A} = \frac{c}{A} - \delta \tag{81}
$$

Let $\frac{c(t)}{A(t)} = y(t)$. Then we have $c(t) = A(t)y(t)$ and $\frac{\dot{c}}{c} = \frac{\dot{A}}{A} + \frac{\dot{y}}{y}$. Substituting them into (80) and (81) leads to

$$
\frac{\dot{y}}{y} = \frac{\beta}{\alpha}y - \frac{1}{1+\alpha}(\beta\delta + \rho + \delta) - \frac{\dot{A}}{A} \tag{82}
$$

$$
\frac{\dot{A}}{A} = y - \delta \tag{83}
$$

Substituting (83) into (82), we have

$$
\begin{aligned}
\frac{\dot{y}}{y} &= \tfrac{\beta}{\alpha}y - \tfrac{1}{1+\alpha}(\beta\delta + \rho + \delta) - (y - \delta) \\
&= \tfrac{\beta-\alpha}{\alpha}y + \delta - \tfrac{1}{1+\alpha}(\beta\delta + \rho + \delta)
\end{aligned}
\tag{84}
$$

Then

$$
\dot{y} \;= [\delta - \frac{1}{1+\alpha}(\beta\delta + \rho + \delta)]y + \frac{\beta-\alpha}{\alpha}y^2 \tag{85}
$$

Notice that (85) is a Bernoulli equation. It can be changed into the following linear ODE after a transformation of $z(t) = \frac{1}{y(t)}$.

$$
\dot{z} = [\frac{1}{1+\alpha}(\beta\delta + \rho + \delta) - \delta]z - \frac{\beta-\alpha}{\alpha} \tag{86}
$$

Denote $\frac{1}{1+\alpha}(\beta\delta + \rho + \delta) - \delta$ and $\frac{\beta-\alpha}{\alpha}$ by m_1 and m_2, respectively. Then (86) becomes

$$\dot{z} = m_1 z - m_2, \quad z(0) = z_0 \tag{87}$$

Then its solution is

$$z(t) = e^{m_1 t}(z_0 - \int_0^t e^{-m_1 s} m_2 ds) = e^{m_1 t}[z_0 + \frac{m_2}{m_1}(e^{-m_1 t} - 1)] \tag{88}$$

Notice the initial value $z_0 = \frac{1}{y(0)} = \frac{1}{c(0)/A(0)} = \frac{A_0}{c_0}$. Then

$$z(t) = e^{m_1 t}[\frac{A_0}{c_0} + \frac{m_2}{m_1}(e^{-m_1 t} - 1)] \tag{89}$$

Then

$$\frac{c}{A} = y(t) = \frac{1}{z(t)} = \frac{1}{e^{m_1 t}[\frac{A_0}{c_0} + \frac{m_2}{m_1}(e^{-m_1 t} - 1)]} \tag{90}$$

Substituting (90) into (83) yields

$$\frac{\dot{A}}{A} = \frac{1}{e^{m_1 t}[\frac{A_0}{c_0} + \frac{m_2}{m_1}(e^{-m_1 t} - 1)]} - \delta \tag{91}$$

Solving (91), we have (see the next Appendix for the computation detail)

$$A(t) = A_0\{\frac{e^{m_1 t}[1 + \frac{m_2}{m_1}/(\frac{A_0}{c_0} - \frac{m_2}{m_1})]}{e^{m_1 t} + \frac{m_2}{m_1}/(\frac{A_0}{c_0} - \frac{m_2}{m_1})}\}^{\frac{1}{m_2}} e^{-\delta t} = A_0\{\frac{\frac{A_0}{c_0}e^{m_1 t}}{\frac{m_2}{m_1} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{m_2}{m_1})}\}^{\frac{1}{m_2}} e^{-\delta t} \tag{92}$$

If we let $c_0 = \eta A_0 = \frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)A_0$, i.e.,

$$\frac{A_0}{c_0} = \frac{1}{\frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha} + \delta)} \tag{93}$$

Noticing the definitions of m_1 and m_2, we have

$$\begin{aligned}\frac{m_2}{m_1} &= \frac{\frac{\beta-\alpha}{\alpha}}{\frac{1}{1+\alpha}(\beta\delta+\rho+\delta)-\delta} = \frac{\frac{\beta-\alpha}{\alpha}}{\frac{\beta\delta+\rho+\delta-(1+\alpha)\delta}{1+\alpha}} \\ &= \frac{\frac{\beta-\alpha}{\alpha}}{\frac{(\beta-\alpha)\delta+\rho}{1+\alpha}} = \frac{1}{\frac{\alpha}{1+\alpha}\frac{(\beta-\alpha)\delta+\rho}{\beta-\alpha}} = \frac{1}{\frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha}+\delta)} \\ &= \frac{1}{\eta} = \frac{A_0}{c_0}\end{aligned} \tag{94}$$

Substituting (94) into (92) leads to

$$A(t) = A_0(e^{m_1 t})^{\frac{1}{m_2}} e^{-\delta t} = A_0 e^{(\eta-\delta)t} \tag{95}$$

Obviously, it is exactly the expression for the optimal addictive capital. By $\frac{m_2}{m_1} = \frac{1}{\eta}$, the general solution for addictive capital and (90) can be rewritten as

$$A(t) = A_0[\frac{\frac{A_0}{c_0}e^{m_1 t}}{\frac{1}{\eta} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{1}{\eta})}]^{\frac{1}{m_2}} e^{-\delta t} \tag{96}$$

$$\frac{c(t)}{A(t)} = \frac{1}{e^{m_1 t}[\frac{A_0}{c_0} + \frac{1}{\eta}(e^{-m_1 t} - 1)]} = \frac{1}{\frac{1}{\eta} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{1}{\eta})} \tag{97}$$

Then

$$c(t) = A(t)\frac{c(t)}{A(t)} = A_0[\frac{\frac{A_0}{c_0}e^{m_1 t}}{\frac{1}{\eta} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{1}{\eta})}]^{\frac{1}{m_2}} \frac{1}{\frac{1}{\eta} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{1}{\eta})} e^{-\delta t} \tag{98}$$

$\square$

J Part II of the Proof of Theorem 2

Let $\ln A(t) = w(t)$. Then $w(0) = w_0 = \ln A_0$, $\frac{\dot{A}}{A} = \dot{w}$ and (91) becomes

$$\dot{w} = \frac{1}{e^{m_1 t}[\frac{A(0)}{c(0)} + \frac{m_2}{m_1}(e^{-m_1 t} - 1)]} - \delta \tag{99}$$

Then

$$\begin{aligned} w(t) &= w_0 + \int_0^t \{\frac{1}{e^{m_1 s}[\frac{A_0}{c_0} + \frac{m_2}{m_1}(e^{-m_1 s} - 1)]} - \delta\}dt \\ &= w_0 + \int_0^t \frac{1}{e^{m_1 s}[\frac{A_0}{c_0} + \frac{m_2}{m_1}(e^{-m_1 s} - 1)]}dt - \delta t \\ &= w_0 + \int_0^t \frac{1}{(\frac{A_0}{c_0} - \frac{m_2}{m_1})e^{m_1 s} + \frac{m_2}{m_1}}dt - \delta t \\ &= w_0 + \int_0^t \frac{1}{h_1 e^{m_1 s} + h_2}dt - \delta t \end{aligned} \tag{100}$$

In the last line of (100), we have already assumed $h_1 = \frac{A_0}{c_0} - \frac{m_2}{m_1}$ and $h_2 = \frac{m_2}{m_1}$. Further, let $x(t) = e^{m_1 t}$. Then $t = \frac{1}{m_1} \ln x$ and $dt = \frac{1}{m_1}\frac{1}{x}dx$. Substituting them into (100) gives

$$
\begin{aligned}
w(t) &= w_0 + \int\limits_0^t \tfrac{1}{h_1 e^{m_1 s} + h_2} dt - \delta t \\
&= w_0 + \tfrac{1}{m_1} \int\limits_1^{e^{m_1 t}} \tfrac{1}{h_1 x + h_2} \tfrac{1}{x} dx - \delta t \\
&= w_0 + \tfrac{1}{m_1} \int\limits_1^{e^{m_1 t}} \tfrac{1}{h_1 (x + h_2/h_1)} \tfrac{1}{x} dx - \delta t \\
&= w_0 + \tfrac{1}{m_1 h_1} \int\limits_1^{e^{m_1 t}} \tfrac{1}{x + h_2/h_1} \tfrac{1}{x} dx - \delta t \\
&= w_0 + \tfrac{1}{m_1 h_1} \int\limits_1^{e^{m_1 t}} \tfrac{1}{h_2/h_1} (\tfrac{1}{x} - \tfrac{1}{x + h_2/h_1}) dx - \delta t \\
&= w_0 + \tfrac{1}{m_1 h_2} \int\limits_1^{e^{m_1 t}} (\tfrac{1}{x} - \tfrac{1}{x + h_2/h_1}) dx - \delta t \\
&= w_0 + \tfrac{1}{m_1 h_2} [\ln x - \ln(x + h_2/h_1)]|_1^{e^{m_1 t}} - \delta t \\
&= w_0 + \tfrac{1}{m_1 h_2} (\ln \tfrac{x}{x + h_2/h_1})|_1^{e^{m_1 t}} - \delta t \\
&= w_0 + \tfrac{1}{m_1 h_2} (\ln \tfrac{e^{m_1 t}}{e^{m_1 t} + h_2/h_1} - \ln \tfrac{1}{1 + h_2/h_1}) - \delta t \\
&= w_0 + \tfrac{1}{m_1 h_2} \ln \tfrac{e^{m_1 t}(1 + h_2/h_1)}{e^{m_1 t} + h_2/h_1} - \delta t
\end{aligned}
\tag{101}
$$

From $\ln A(t) = w(t)$ and (101), we have

$$
\begin{aligned}
A(t) &= \exp(w(t)) = e^{[w_0 + \frac{1}{m_1 h_2} \ln \frac{e^{m_1 t}(1 + h_2/h_1)}{e^{m_1 t} + h_2/h_1} - \delta t]} \\
&= A_0 [\tfrac{e^{m_1 t}(1 + h_2/h_1)}{e^{m_1 t} + h_2/h_1}]^{\frac{1}{m_1 h_2}} e^{-\delta t} \\
&= A_0 \{\tfrac{e^{m_1 t}[1 + \frac{m_2}{m_1}/(\frac{A_0}{c_0} - \frac{m_2}{m_1})]}{e^{m_1 t} + \frac{m_2}{m_1}/(\frac{A_0}{c_0} - \frac{m_2}{m_1})}\}^{\frac{1}{m_2}} e^{-\delta t} \\
&= A_0 [\tfrac{\frac{A_0}{c_0} e^{m_1 t}}{\frac{m_2}{m_1} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{m_2}{m_1})}]^{\frac{1}{m_2}} e^{-\delta t}
\end{aligned}
\tag{102}
$$

□

K The Computation of t^*

Let $\varepsilon > 0$ and $c_0 = (\eta + \varepsilon) A_0$. Substituting it into (102) leads to

$$
A(t) = A_0 (\frac{\frac{1}{\eta + \varepsilon}}{\frac{1}{\eta} e^{-m_1 t} + \frac{1}{\eta + \varepsilon} - \frac{1}{\eta}})^{\frac{1}{m_2}} e^{-\delta t} = A_0 [\frac{\frac{1}{\eta + \varepsilon}}{\frac{1}{\eta} e^{-m_1 t} - \frac{\varepsilon}{\eta(\eta + \varepsilon)}}]^{\frac{1}{m_2}} e^{-\delta t}
$$

If the denominator $\frac{1}{\eta} e^{-m_1 t} - \frac{\varepsilon}{\eta(\eta + \varepsilon)} \to 0$, then $A \to \infty$. In fact, it suffices to make the denominator $\frac{1}{\eta} e^{-m_1 t} - \frac{\varepsilon}{\eta(\eta + \varepsilon)} \to 0$ if $t \to t^* = -\frac{1}{m_1} \ln \frac{\varepsilon}{\eta + \varepsilon}$. □

L Objective Value

$$
\begin{aligned}
&\int_0^\infty -\frac{A^\beta}{c^\alpha}e^{-\rho t}dt = \int_0^\infty -\frac{A^\alpha}{c^\alpha}A^{\beta-\alpha}e^{-\rho t}dt \\
&= -\int_0^\infty [\frac{1}{\eta} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{1}{\eta})]^\alpha \{A_0[\frac{\frac{A_0}{c_0}e^{m_1 t}}{\frac{1}{\eta}+e^{m_1 t}(\frac{A_0}{c_0}-\frac{1}{\eta})}]^{\frac{1}{m_2}}e^{-\delta t}\}^{\beta-\alpha}e^{-\rho t}dt \\
&= -\int_0^\infty A_0^{\beta-\alpha}[\frac{1}{\eta} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{1}{\eta})]^\alpha [\frac{\frac{A_0}{c_0}e^{m_1 t}}{\frac{1}{\eta}+e^{m_1 t}(\frac{A_0}{c_0}-\frac{1}{\eta})}]^{\frac{\beta-\alpha}{m_2}}e^{-[(\beta-\alpha)\delta+\rho]t}dt \\
&= -\int_0^\infty A_0^{\beta-\alpha}[\frac{1}{\eta} + e^{m_1 t}(\frac{A_0}{c_0} - \frac{1}{\eta})]^\alpha [\frac{\frac{A_0}{c_0}e^{m_1 t}}{\frac{1}{\eta}+e^{m_1 t}(\frac{A_0}{c_0}-\frac{1}{\eta})}]^{\alpha}e^{-[(\beta-\alpha)\delta+\rho]t}dt \\
&= -\int_0^\infty A_0^{\beta-\alpha}(\frac{A_0}{c_0}e^{m_1 t})^\alpha e^{-[(\beta-\alpha)\delta+\rho]t}dt \\
&= -A_0^{\beta-\alpha}(\frac{A_0}{c_0})^\alpha \int_0^\infty e^{\alpha m_1 t}e^{-[(\beta-\alpha)\delta+\rho]t}dt \\
&= -A_0^{\beta-\alpha}(\frac{A_0}{c_0})^\alpha \int_0^\infty e^{\{\frac{\alpha}{1+\alpha}[(\beta-\alpha)\delta+\rho]-(\beta-\alpha)\delta+\rho\}t}dt \\
&= -A_0^{\beta-\alpha}(\frac{A_0}{c_0})^\alpha \int_0^\infty e^{-\frac{1}{1+\alpha}[\rho+(\beta-\alpha)\delta]t}dt \\
&= -A_0^{\beta-\alpha}(\frac{A_0}{c_0})^\alpha \frac{1+\alpha}{\rho+(\beta-\alpha)\delta}
\end{aligned}
\tag{103}
$$

Recall that $\frac{A_0}{c_0} = \frac{1}{\eta}$ corresponds to the optimal initial condition. Substituting it into (103) leads to

$$
\begin{aligned}
&-A_0^{\beta-\alpha}(\frac{A_0}{c_0})^\alpha \frac{1+\alpha}{\rho+(\beta-\alpha)\delta} \\
&= -A_0^{\beta-\alpha}(\frac{1}{\eta})^\alpha \frac{1+\alpha}{\rho+(\beta-\alpha)\delta} \\
&= -A_0^{\beta-\alpha}[\frac{1}{\frac{\alpha}{1+\alpha}(\frac{\rho}{\beta-\alpha}+\delta)}]^\alpha \frac{1+\alpha}{\rho+(\beta-\alpha)\delta} \\
&= -A_0^{\beta-\alpha}\frac{(1+\alpha)^{1+\alpha}}{[\rho+(\beta-\alpha)\delta][\alpha(\frac{\rho}{\beta-\alpha}+\delta)]^\alpha} \\
&= -A_0^{\beta-\alpha}\{\frac{(1+\alpha)}{[\rho+(\beta-\alpha)\delta]^{\frac{1}{1+\alpha}}[\alpha(\frac{\rho}{\beta-\alpha}+\delta)]^{\frac{\alpha}{1+\alpha}}}\}^{1+\alpha} \\
&= -A_0^{\beta-\alpha}\{\frac{(1+\alpha)}{[\rho+(\beta-\alpha)\delta]^{\frac{1}{1+\alpha}}[\alpha\frac{\rho+(\beta-\alpha)\delta}{\beta-\alpha}]^{\frac{\alpha}{1+\alpha}}}\}^{1+\alpha} \\
&= -A_0^{\beta-\alpha}\{\frac{(1+\alpha)}{\alpha^{\frac{\alpha}{1+\alpha}}[\rho+(\beta-\alpha)\delta]^{\frac{1}{1+\alpha}}\frac{[\rho+(\beta-\alpha)\delta]^{\frac{\alpha}{1+\alpha}}}{(\beta-\alpha)^{\frac{\alpha}{1+\alpha}}}}\}^{1+\alpha} \\
&= -A_0^{\beta-\alpha}\{\frac{(1+\alpha)(\beta-\alpha)^{\frac{\alpha}{1+\alpha}}}{\alpha^{\frac{\alpha}{1+\alpha}}[\rho+(\beta-\alpha)\delta]}\}^{1+\alpha} \\
&= -A_0^{\beta-\alpha}\{\frac{(1+\alpha)(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}}}{[\rho+(\beta-\alpha)\delta]}\}^{1+\alpha} \\
&= -A_0^{\beta-\alpha}\{\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}}+\alpha(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}}}{[\rho+(\beta-\alpha)\delta]}\}^{1+\alpha} \\
&= -A_0^{\beta-\alpha}\{\frac{(\frac{\beta-\alpha}{\alpha})^{\frac{\alpha}{1+\alpha}}+(\beta-\alpha)^{\frac{\alpha}{1+\alpha}}\alpha^{\frac{1}{1+\alpha}}}{[\rho+(\beta-\alpha)\delta]}\}^{1+\alpha}
\end{aligned}
\tag{104}
$$

(104) is identical to the one obtained by HJB method if we set $\sigma = 0$. □