Competing over a Finite Number of Locations

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Abstract

We consider spatial competition when consumers are arbitrarily distributed on a compact metric space. Retailers can choose one of finitely many locations in this space. We first prove that a pure strategy equilibrium exists if the number of retailers is large enough, while it need not exist for a small number of retailers. Symmetric mixed equilibria exist for any number of retailers. We then prove that the distribution of retailers tends to agree with the distribution of the consumers both at the pure strategy equilibrium and at the symmetric mixed one. The results are shown to be robust to the introduction of (i) randomness in the number of retailers and (ii) different ability of the retailers to attract consumers.

JEL Classification: C72, R30, R39.
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1 Introduction

Consider a market with consumers and retailers. Consumers are distributed on the unit interval and each one of them shops at the closest store. Retailers choose their location in order to achieve the largest fraction of consumers. This model is called the Pure Location Game and was initially considered by Hotelling (1929) for the case of two retailers. This seminal paper has been extended and applied in different fields such as industrial organization or spatial competition, giving rise to an immense literature. Downs (1957) used Hotelling's model to explain political competition.

As far as the equilibria of this game are concerned, two main regularities appear. First, a pure strategy equilibrium may fail to exist. With at least four retailers locating at the unit interval, there is no pure equilibrium if the consumers' distribution has either a strictly convex or a strictly concave density. This was shown by Osborne and Pitchik (1986, Proposition 2) and is an immediate consequence of a theorem in Eaton and Lipsey (1975).

The second stylized fact is that when the number of retailers becomes large, the location of the retailers at the symmetric mixed equilibrium tends to coincide with the distribution of the consumers on the space. This phenomenon where "retailers match consumers" was first observed by Osborne and Pitchik (1986)[1]. A similar result is present in Laster, Bennet, and Geoum (1999) and Ottaviani and Sorensen (2006) in the context of professional forecasting.

Our paper extends the classical model in two directions. First, we allow the consumers to be distributed on a general, possibly multidimensional, space. This multidimensional space might account for location choices or various characteristics of the product that consumers care about[2]. Our modelling approach is quite general since we assume that the consumers' distribution is arbitrary; an assumption of nonatomicity is made only out of simplicity. Second, we remove the typical assump-

[1]Formally, Osborne and Pitchik (1986) prove that the symmetric equilibrium strategies satisfy the claim assuming that the consumers are distributed in the interval [0,1] according to any twice continuously differentiable distribution function.

[2]Our model can be reinterpreted in terms of political competition. Within the paper, we stick to the interpretation of retailers/consumers for simplicity.

tion that the action space of the retailers coincides with the support of the consumers' distribution. Note that this assumption is not verified in many real-life applications, for instance when zoning regulations are enforced[3]. In this case, specific commercial activities may be allowed only in areas that would not produce negative externalities on the population and guarantee the existence of services, like parking space, etc.. In general there are several cases where the strategic behavior of the retailers is subject to feasibility constraints. From a theoretical point of view, this restriction is present in network economics in which retailers can locate only in some points of a graph, e.g., its vertices. In other words, a particular case of our model are the network-based models.

Building on these extensions, we first consider a simple version of the model, where all retailers are symmetric. First, we prove that, when the number of retailers is large enough, there exists a pure strategy equilibrium. Note that this result does not depend on the underlying distribution of the consumers[4]. In this equilibrium, the distribution of retailers gets closer to the real distribution of consumers as the number of retailers grows. We then examine the properties of symmetric mixed strategy equilibria (which must exist since the game is finite and symmetric). We first prove that, as the number of retailers grows large, every symmetric equilibrium must be completely mixed. In other words, in these equilibria, every feasible location is occupied with positive probability. This implies that the expected payoff from choosing each location must be equal for each retailer. A non-trivial consequence of this is that the distribution of retailers induced by the symmetric mixed equilibrium converges to the consumers' distribution. This shows a strong analogy between the non-symmetric pure equilibria and the symmetric mixed equilibrium.

Once we have considered the simple model with an exogenous number of symmet-

[3]Land use regulation has been extensively analyzed in urban economics, mostly from an applied perspective. It is often argued that zoning can have anti-competitive effects and at the same time be beneficial since it might solve problems of externalities (see Suzuki, 2013, for a recent work on this area.)

[4]For an existence result in networks with uniformly distributed consumers, see Fournier and Scarsini (2014). The existence of pure equilibrium seems to be harder to achieve when prices are taken into account (see Heijnen and Soetevent, 2014, for a recent contribution on networks.)

ric retailers, we then examine two extensions. The first extension deals with games with a random number of players and the second one introduces ex-ante asymmetries between the retailers.

As far as the first extension is concerned, it is well-known that games with a large number of players can easily produce results that are not robust with respect to the number of players. In order to check this robustness, we consider also a model where the number of players is random, using Poisson games à la Myerson (1998, 2000). We show that in the unique equilibrium of the Poisson game retailers match consumers when the parameter of the Poisson distribution is large enough, so retailers do not even need to know the exact number of their competitors to play their (mixed) equilibrium strategies.

Finally, we consider a richer model where the retailers are of two different types, advantaged and disadvantaged. Consumers prefer advantaged retailers, so they are ready to travel a bit more to shop at one of them rather than at a disadvantaged one. Here we model the comparative advantage of the first type of retailers by an additive constant. This is formally equivalent to the idea of valence in election models (see Aragones and Palfrey, 2002, Aragonès and Xefteris, 2012, among others). We show that, when the number of advantaged players increases, they play as if the disadvantaged retailers did not exist, and these ones get a zero payoff, no matter what they do.

In the whole paper we assume that competition among retailers is only in term of location, not price. We do this for several reasons. First, there exist several markets where price is not decided by retailers: think, for instance of newsvendors, shops operating under franchising, pharmacies in many countries, etc.. Second, our model without pricing can be used to study other topics, e.g., political competition, when candidates have to take position on several, possibly related, issues. Finally several of the existing models that allow competition on location and pricing are two-stage models, where competition first happens on location and subsequently on price. Our game could be seen as a model of the first stage. It is interesting to notice that the recent paper by Heijnen and Soetevent (2014) deals with the second stage in a location model on a graph, assuming that the first has already been solved.

Review of the Literature

We refer the reader to Fournier and Scarsini (2014) for a recent survey of the literature on Hotelling games. Here we just mention the articles that are somehow closer to what we do in our paper.

Eaton and Lipsey (1975) consider a Hotelling-type model with an arbitrary number of players, different possible structures of the space where retailers can locate, and different distributions of the customers. Lederer and Hurter (1986) consider a model with two retailers where consumers are non-uniformly distributed on the plane. Aoyagi and Okabe (1993) look at a bidimensional market and, through simulation, relate the existence of equilibria and their properties to the shape of the market. Tabuchi (1994) considers a two-stage Hotelling duopoly model in a bidimensional market. Hörner and Jamison (2012) look at a Hotelling model with a finite number of customers.

Dürr and Thang (2007), Mavronicolas, Monien, Papadopoulou, and Schoppmann (2008), Feldmann, Mavronicolas, and Monien (2009), and Gur, Saban, and Stier-Moses (2014) consider a Hotelling model on graphs where retailers can locate only on the vertices of the graph. Pálvölgyi (2011) and Fournier and Scarsini (2014) consider Hotelling games on graphs with an arbitrary number of players. Heijnen and Soetevent (2014) extend Hotelling's model of price competition with quadratic transportation costs from a line to graphs.

The paper is organized as follows. Section 2 introduces the model. Section 3 analyzes its equilibria. Section 4 considers the case of a random number of retailers. Section 5 deals with the case of differentiated retailers. All proofs are in the Appendix.

2 The model

In this section we describe the basic location model, whose different variations will be studied in the rest of the paper.

Consumers. In this model consumers are distributed according to a measure λ on a compact Borel metric space (S, d). For instance S could be a compact subset of $\mathbb{R}^2$ or a compact subset of a 2-sphere, but it could also be a (properly metrized) network.

Retailers. A finite set $N_n := \{1, \ldots, n\}$ of retailers have to decide where to set shop, knowing that consumers choose the closest retailers. Each retailer wants to maximize her market share. The action set of each retailer is a finite subset of S. This means that, unlike what happens in a typical Hotelling-type model, retailers cannot locate anywhere they want, but can choose only one of finitely many possible locations. For instance they can set shop only in one of the existing shopping malls in town.

Tessellation. More formally, define $K = \{1, \ldots, k\}$ and let $X_K := \{x_1, \ldots, x_k\} \subset S$ be a finite collection of points in S. These are the points where retailers can open a store. For every $J \subset K$ call $X_J := \{x_j : j \in J\}$ and consider the Voronoi tessellation $V(X_J)$ of S induced by X_J. That is, for each $x_j \in X_J$ define the Voronoi cell of x_j as follows:

$$v_J(x_j) := \{y \in S : d(y, x_j) \leq d(y, x_\ell) \text{ for all } x_\ell \in X_J\}.$$

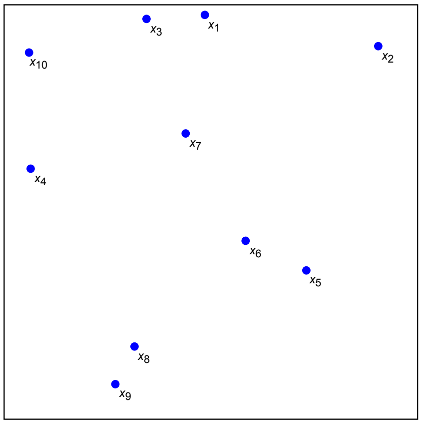
The cell $v_J(x_j)$ contains all points whose distance from x_j is not larger than the distance from the other points in X_J. Call

$$V(X_J) := (v_J(x_j))_{j \in J}$$

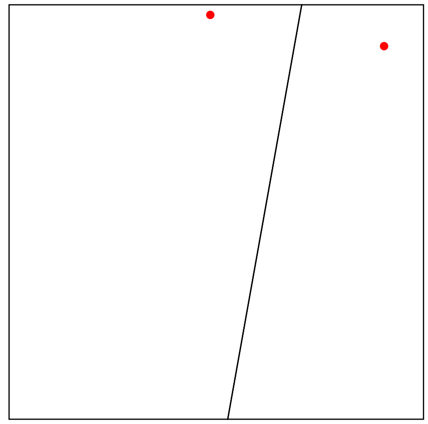
the set of all Voronoi cells $v_J(x_j)$. See, for instance, Figure 1. It is clear that for $J \subset L \subset K$ we have $v_J(x_j) \supset v_L(x_j)$ for every $j \in J$.

FIGURE 1 ABOUT HERE

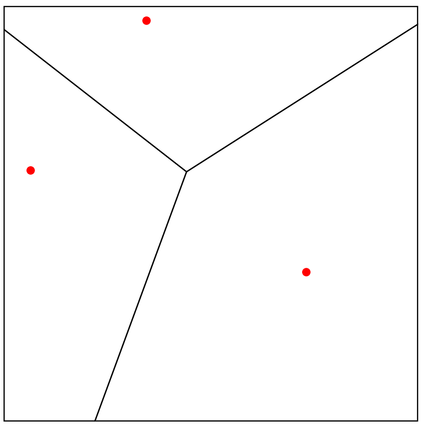
Given that λ is the distribution of consumers on the space S, we have that $\lambda(v_J(x_j))$ is the mass of consumers who are weakly closer to x_j than to any other



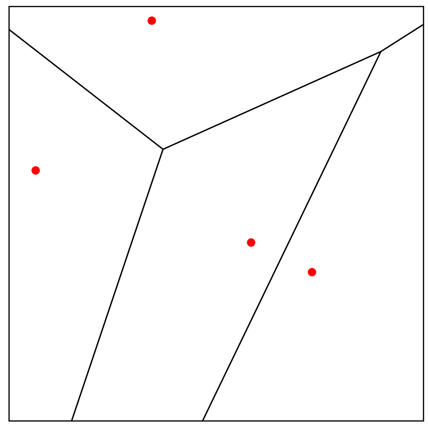
(a) $X_K \subset [0,1]^2,\ K = \{1,\ldots,10\}$.



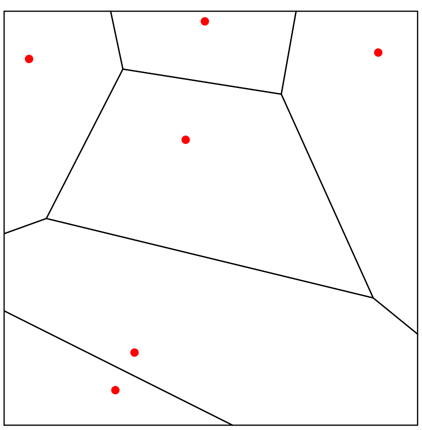
(b) $V(X_J),\ J = \{1,2\}$.



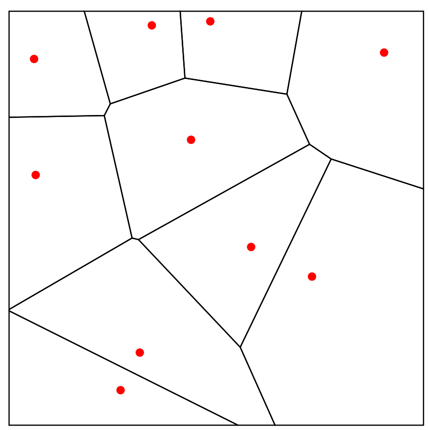
(c) $V(X_J),\ J = \{3,4,5\}$.



(d) $V(X_J),\ J = \{3,4,5,6\}$.



(e) $V(X_J),\ J = \{1,2,7,8,9,10\}$.



(f) $V(X_J),\ J = K$.

Figure 1: Various Voronoi tessellations with different subsets of locations

point in X_J. If price is homogeneous, these consumers will prefer to shop at location x_j rather than at other locations in X_J. Consumers that belong to r different Voronoi cells $v_J(x_{j_1}), \dots, v_J(x_{j_r})$, are equally likely to shop at any of the locations $x_{j_1}, \dots, x_{j_r}$. To simplify the notation and the results, we assume that S is a compact subset of some Euclidean space, that λ is absolutely continuous with respect to the Lebesgue measure on this space and

$$\lambda(v_K(x_j)) > 0 \quad \text{for all } x_j \in X_K. \tag{2.1}$$

More general situations can be considered but they require more care in handling ties.

The game. We will build a game where $N_n := \{1, \dots, n\}$ is the set of players. For $i \in N_n$ call $a_i \in X_K$ the action of player i. Then $\boldsymbol{a} := (a_i)_{i \in N_n}$ is the profile of actions and $\boldsymbol{a}_{-i} := (a_h)_{h \in N_n \setminus \{i\}}$ is the profile of actions of all the players different from i. Hence $\boldsymbol{a} = (a_i, \boldsymbol{a}_{-i})$.

We say that $\boldsymbol{a} := (a_1, \dots, a_n) \approx X_J$ if for all locations $x_j \in X_J$ there exists a player $i \in N_n$ such that $a_i = x_j$ and for all players $i \in N_n$ there exists a location $x_j \in X_J$ such that $a_i = x_j$.

For $i \in N_n$, the payoff of player i is $u_i : X_K^n \to \mathbb{R}$, defined as follows:

$$u_i(\boldsymbol{a}) = \frac{1}{\operatorname{card}\{h : a_h = a_i\}} \sum_{J \subset K} \lambda(v_J(a_i)) \mathbb{1}(\boldsymbol{a} \approx X_J). \tag{2.2}$$

The idea behind expression (2.2) is the following. Player i's payoff is the measure of the consumers that are closer to the location that she chooses than to any other location chosen by any other player, divided by the number of retailers that choose the same action as i. As Figure 1 shows, some locations may not be chosen by any player, this is why, for every $J \subset K$, we have to consider the Voronoi tessellation $V(X_J)$ with $\boldsymbol{a} \approx X_J$ rather than the finer tessellation $V(X_K)$. We examine a simple example to clarify the idea.

Example 2.1. Let $S = [0, 1]$, let λ be the Lebesgue measure on $[0, 1]$, and let $X_K = \{0, 1/2, 1\}$. As mentioned before, for any given X_J, the Voronoi cell of location x_j

represents the set of points in $[0,1]$ that are closer to x_j than any other point in X_J.

$$v_J(0) = \begin{cases} [0,1] & \text{if } X_J = \{0\}, \\ [0,1/2] & \text{if } X_J = \{0,1\}, \\ [0,1/4] & \text{if } X_J = X_K \text{ or } X_J = \{0,1/2\}. \end{cases}$$

$$v_J(1/2) = \begin{cases} [0,1] & \text{if } X_J = \{1/2\}, \\ [1/4,1] & \text{if } X_J = \{0,1/2\} \\ [0,3/4] & \text{if } X_J = \{1/2,1\}, \\ [1/4,3/4] & \text{if } X_J = X_K. \end{cases}$$

$$v_J(1) = \begin{cases} [0,1] & \text{if } X_J = \{1\}, \\ [1/2,1] & \text{if } X_J = \{0,1\}, \\ [3/4,1] & \text{if } X_J = X_K \text{ or } X_J = \{1/2,1\}. \end{cases}$$

Hence

$$\lambda(v_J(0)) = \begin{cases} 1 & \text{if } X_J = \{0\}, \\ 1/2 & \text{if } X_J = \{0,1\}, \\ 1/4 & \text{if } X_J = X_K \text{ or } X_J = \{0,1/2\}. \end{cases}$$

$$\lambda(v_J(1/2)) = \begin{cases} 1 & \text{if } X_J = \{1/2\}, \\ 3/4 & \text{if } X_J = \{0,1/2\} \text{ or } X_J = \{1/2,1\}, \\ 1/2 & \text{if } X_J = X_K. \end{cases}$$

$$\lambda(v_J(1)) = \begin{cases} 1 & \text{if } X_J = \{1\}, \\ 1/2 & \text{if } X_J = \{0,1\}, \\ 1/4 & \text{if } X_J = X_K \text{ or } X_J = \{1/2,1\}. \end{cases}$$

Therefore the payoff for player i, if she chooses location 0 when the rest of the players' pure actions are $\boldsymbol{a}_{-i}$ is

$$u_i(0, \boldsymbol{a}_{-i}) = \frac{1}{\operatorname{card}\{h : a_h = a_i\}} \phi(\boldsymbol{a}_{-i}),$$

where

$$\phi(\boldsymbol{a}_{-i}) = \begin{cases} 1 & \text{if } \boldsymbol{a} \approx \{0\}, \\ \frac{1}{2} & \text{if } \boldsymbol{a} \approx \{0, 1\}, \\ \frac{1}{4} & \text{if } \boldsymbol{a} \approx X_K \text{ or } \boldsymbol{a} \approx \{0, 1/2\}. \end{cases}$$

The payoffs when she chooses either 1/2 or 1 can be similarly computed.

Remark 2.2. As mentioned before, the total demand for a location x_j (i.e. share of consumers that purchase the good from a given location) depends on the location of all the retailers. The minimum value that this demand can assume is equal to $\lambda(v_K(x_j)) > 0$, which happens when there is at least one retailer in each location (i.e. when $\boldsymbol{a} \approx X_K$). This represents one of the main differences with respect to the classical model in which retailers can locate everywhere in the set S. In the classical model the demand for a location could be made arbitrarily small. To see why, consider the classical Downsian model in the interval $[0, 1]$ with three players. Assume, for instance that player 1 locates in x, player 2 locates in $x - \varepsilon$ and player 3 locates in $x + \varepsilon$. Then the total demand for x can be rendered arbitrary small as $\varepsilon \to 0$.

Consider a game where the consumers are distributed on S according to λ, the set of players is N_n, the set of actions for each player is X_K and the payoff of player i is given by (2.2). Call this game $\mathscr{G}_n = \langle S, \lambda, N_n, X_K, (u_i) \rangle$. Since the set of actions coincides with the set of locations, we will use the two terms interchangeably.

With an abuse of notation, we use the same symbol $\mathscr{G}_n$ for the mixed extension of the game, where, for a mixed strategy profile $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$, the expected payoff of player i is

$$U_i(\boldsymbol{\sigma}) = \sum_{a_1 \in X_K} \cdots \sum_{a_n \in X_K} u_i(\boldsymbol{a}) \sigma_1(a_1) \dots \sigma_n(a_n).$$

3 Equilibria

In the rest of this section, unless otherwise stated, we consider a sequence $\{\mathscr{G}_n\}$ of games, all of which have the same parameters S, λ, X_K. More precisely, our focus is on the sequence of games when the number of retailers n grows.

We prove that when the number of retailers is large enough, (i) the game admits a pure strategy equilibrium and (ii) the distribution of retailers in equilibrium approaches the distribution of consumers both in the pure strategy equilibrium and in the symmetric mixed one.

3.1 Pure equilibria

We now show that, when the game is large, the game $\mathscr{G}_n$ admits pure equilibria. As the next example shows, this result does not hold when the number of players is small.

Example 3.1 (A game without pure equilibria)**.** Consider a game $\mathscr{G}_n$ with $n = 3$, $S = [0, 1]$, λ the Lebesgue measure, and $X_K = \{i/100 : i = 0, \ldots, 100\}$. Assume that the game admits a pure strategy equilibrium (a_1, a_2, a_3), where, without any loss of generality, $a_1 \leq a_2 \leq a_3$. We see that, if $a_1 < a_2 - 1/100$, then player 1 has an incentive to deviate to $a_2 - 1/100$; if $a_2 + 1/100 < a_3$, then player 3 has an incentive to deviate to $a_2 + 1/100$. The profile $a_1 = a_2 - 1/100$ and $a_3 = a_2 + 1/100$ is not an equilibrium, because player 2 can deviate to either $a_1 - 1/100$ or to $a_3 + 1/100$ and at least one of the deviations is profitable. If $a_1 = a_2$ and $a_3 > a_1 + 1/100$, then player 3 can profitably deviate to $a_1 + 1/100$. If $a_1 = a_2$ and $a_3 = a_1 + 1/100$, then either player 3 can profitably deviate to $a_1 - 1/100$ or player 1 can profitably deviate to $a_3 + 1/100$. A similar argument excludes $a_1 < a_2 = a_3$. Finally, if $a_1 = a_2 = a_3$, player 1 can deviate to either $a_1 - 1/100$ or $a_1 + 1/100$ and at least one deviation is profitable. This proves by contradiction that there is no pure strategy equilibrium in this game.

Example 3.2 (Weakly dominated locations)**.** Consider a game $\mathscr{G}_n$ with $n = 2$, $S =$

$[0,1]$, λ the Lebesgue measure, and $X_K = \{0.45, 0.5, 0.55\}$. Then both 0.45 and 0.55 are weakly dominated by 0.5.

The existence of weakly dominated strategies becomes impossible when the number of players is large enough.

Proposition 3.3. *Consider a sequence of games $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$. There exists $\bar{n}$ such that for all $n \geq \bar{n}$ no location in X_K is weakly dominated.*

When the number of players is large, pure equilibria exist and the share of players in the different locations in equilibrium is approximately proportional to the measure of the corresponding Voronoi cells. The following theorem makes this idea precise.

Theorem 3.4. *Consider a sequence of games $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$. There exists $\bar{n}$ such that for all $n \geq \bar{n}$ the game $\mathscr{G}_n$ admits a pure equilibrium $\boldsymbol{a}^*$. Moreover, for all $n \geq \bar{n}$, any pure equilibrium is such that*

$$\frac{n_j(\boldsymbol{a}^*)}{n_\ell(\boldsymbol{a}^*)+1} \leq \frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))} \leq \frac{n_j(\boldsymbol{a}^*)+1}{n_\ell(\boldsymbol{a}^*)}. \tag{3.1}$$

3.2 Mixed equilibria

We now consider the mixed equilibria of the game $\mathscr{G}_n$.

Theorem 3.5. *For every $n \in \mathbb{N}$ the game $\mathscr{G}_n$ admits a symmetric mixed equilibrium $\boldsymbol{\gamma}^{(n)} = (\gamma^{(n)}, \dots, \gamma^{(n)})$ such that*

$$\lim_{n\to\infty} \gamma^{(n)} = \gamma, \tag{3.2}$$

with

$$\gamma(x_j) = \frac{\lambda(v_K(x_j))}{\lambda(S)} \quad \text{for all } j \in K. \tag{3.3}$$

We can easily prove that (3.2) holds only asymptotically. For instance consider a game $\mathscr{G}_n$ with $n = 2$, $S = [0,1]$, λ the Lebesgue measure, and $X_K = \{0.45, 0.5, 0.55\}$. Then the only symmetric equilibrium is the pure profile where both players choose the location 0.5.

Similarly, let $S = [0, 1]$ with λ the Lebesgue measure on $[0, 1]$ and $X_K = \{0, 0.5, 1\}$. For each $n > 3$, the game $\mathscr{G}_n$ admits a symmetric mixed equilibrium $\boldsymbol{\gamma}^{(n)}$, where

$$\gamma^{(n)}(0) = \gamma^{(n)}(1) = p_n, \quad \gamma^{(n)}(0.5) = 1 - 2p_n,$$

with p_n as follows:

n	4	5	6	7	8	9	10	15	20
p_n	0.113	0.167	0.196	0.214	0.225	0.232	0.237	0.247	0.249

As shown by the table, the probabilities in the symmetric mixed equilibrium converge towards the ones described by Theorem 3.5.

It is interesting to notice that the outcome of pure equilibria mimics the expected outcome of the mixed equilibria. In other words, the number of players who choose an action in a pure equilibrium is close to the expected number of players who choose the same action in the symmetric mixed equilibrium. Obviously no pure equilibrium can be symmetric.

4 Games with a random number of players

In this section we consider games where the number of players is random and we show how the results of the previous section extend to this case. In particular we focus on Poisson games (see Myerson, 1998, 2000, among others). In these games, the number of players follows a Poisson distribution. We call $\mathscr{P}_n = \langle S, \lambda, N_{\Xi_n}, X_K, (u_i) \rangle$ the game where the cardinality of the players set N_{Ξ_n} is a random variable Ξ_n, with

$$\mathbb{P}(\Xi_n = k) = \frac{\mathrm{e}^{-n}\, n^k}{k!},$$

that is, Ξ_n has a Poisson distribution with parameter n.

Just like in game $\mathscr{G}_n$, in game $\mathscr{P}_n$ all players have the same utility function. So the utility function of player i depends only on i' s action and on the number of players who have chosen x_j for all $j \in K$.

Quoting Myerson (1998), "population uncertainty forces us to treat players symmetrically in our game-theoretic analysis," so each player choses action x_j with probability $\sigma(x_j)$. As a consequence, all equilibria are symmetric. Properties of the Poisson distribution imply that the number of players choosing action x_j is independent of the number of players choosing action x_ℓ for $j \neq \ell$.

The expected utility of each player, when she chooses action x_j and all the other players act according to the mixed action σ is

$$U(x_j, \sigma) = \sum_{y \in Z(X_K)} \prod_{j \in K} \left(\frac{\mathrm{e}^{-n\sigma(x_j)} (n\sigma(x_j))^{y(x_j)}}{y(x_j)} \right) U(x_j, y),$$

where $Z(X_K)$ denotes the set of possible action profiles for the players in a Poisson game. That is, $Z(X_K)$ is the set of vectors $y = (y(x_i))_{x_i \in X_K}$ such that each component $y(x_i)$ is a nonnegative integer that describes the number of players choosing action x_i.

In the rest of this section we consider a sequence $\{\mathscr{P}_n\}$ of games, all of which have the same parameters S, λ, X_K.

Theorem 4.1. *For every $n \in \mathbb{N}$ the game $\mathscr{P}_n$ admits a symmetric equilibrium $\gamma^{(n)}$ such that*

$$\lim_{n \to \infty} \gamma^{(n)}(x_j) = \frac{\lambda(v_K(x_j))}{\lambda(S)} \quad \textit{for all } j \in K. \tag{4.1}$$

The next example shows that in general the equilibria of $\mathscr{G}_n$ and $\mathscr{P}_n$ do not coincide.

Example 4.2. Let $S = [0, 1]$ with λ the Lebesgue measure on $[0, 1]$ and $X_K = \{0.1, 0.5, 0.9\}$. We consider the equilibria of the games $\mathscr{G}_3$ (static) and $\mathscr{P}_3$ (Poisson).

In the game $\mathscr{G}_3$, there exists an equilibrium σ^* in which each retailer locates in 0.5. Under σ^* the payoff for each retailer equals $1/3$ since they uniformly split the consumers in S. A deviation towards 0.1 or 0.9 would give a payoff of $0.3 < 1/3$, so σ^* is indeed an equilibrium of $\mathscr{G}_3$.

We now prove that σ^* is not an equilibrium in the game $\mathscr{P}_3$. We have

$$U(\sigma^*) = \frac{1 - \mathrm{e}^{-3}}{3} \approx 0.316738,$$
$$U(0.1, \sigma^*) = U(0.9, \sigma^*) = \mathrm{e}^{-3} + (0.3)(1 - \mathrm{e}^{-3}) \approx 0.334851.$$

This shows that a deviation to either 0.1 or 0.9 is profitable, hence σ^* is not an equilibrium of the game $\mathscr{P}_3$.

5 Competition with different classes of retailers

Up to now, we have considered a model where all retailers are equally able to attract consumers. In other words, a consumer is indifferent between purchasing the good at two different shops if they are equally distant from her location.

In many situations some retailers have a comparative advantage due, for instance, to reputation. Therefore, *ceteris paribus*, a consumer may prefer one retailer over another. Similar models have been studied in the political competition literature with few strategic parties (see Aragones and Palfrey, 2002, among others). In this literature the term "valence" is used to indicate the competitive advantage of one candidate over another.

In the model that we analyze below, retailers can be of two types: advantaged (A) and disadvantaged (D). We choose this dichotomic model out of simplicity. Results are not qualitatively different when a finite number of types is allowed.

When choosing between two retailers of the same type, a consumer takes into account only their distance from her and she prefers the closer of the two. When choosing between a retailer of type A located in x^A and a retailer of type D located in x^D, a consumer located in y will prefer the retailer of type A iff

$$d(x^A, y) < d(x^D, y) + \beta, \quad \text{with } \beta > 0.$$

She will be indifferent between the two retailers iff

$$d(x^A, y) = d(x^D, y) + \beta.$$

Obviously the case $\beta = 0$ corresponds to the model examined in Section 2.

Different ways to model advantage of one type of players over another have been considered in the literature (see Gouret, Hollard, and Rossignol, 2011, for a discussion).

We now formally define a game $\mathscr{D}_n$ with differentiated retailers. For $j \in \{A, D\}$, call N_n^j the set of retailers of type j and define $n^j = \text{card}(N_n^j)$. Therefore

$$\begin{aligned} N_n &= N_n^A \cup N_n^D, \\ n &= n^A + n^D. \end{aligned}$$

For $j \in \{A, D\}$ and $i \in N_n^j$ call $a_i^j \in X_K$ the action of retailer i. Then the profile of actions is

$$\boldsymbol{a} := (\boldsymbol{a}^A, \boldsymbol{a}^D) := \{(a_i^A)_{i \in N_n^A}, (a_i^D)_{i \in N_n^D}\}.$$

For any profile $\boldsymbol{a} \in X_K^n$ define

$$\begin{aligned} n_j^A(\boldsymbol{a}) &:= \text{card}\{i \in N_n^A : a_i^A = x_j\}, \\ n_j^D(\boldsymbol{a}) &:= \text{card}\{i \in N_n^D : a_i^D = x_j\}. \end{aligned}$$

So n_j^A and n_j^D are the number of A and D players, respectively, who choose action x_j.

We say that $(\boldsymbol{a}^A, \boldsymbol{a}^D) \approx X_{J^A, J^D}$ if for all locations $x_j \in X_{J^A}$ there exists a player $i \in N_n^A$ such that $a_i^A = x_j$ and for all players $i \in N_n^A$ there exists a location $x_j \in X_{J^A}$ such that $a_i^A = x_j$ and for all locations $x_j \in X_{J^D}$ there exists a player $i \in N_n^D$ such that $a_i^D = x_j$ and for all players $i \in N_n^D$ there exists a location $x_j \in X_{J^D}$ such that $a_i^D = x_j$.

Fix $\beta > 0$, and, for $J^A, J^D \subset K$, define

$$\begin{aligned} v^A_{J^A,J^D}(x_j) &:= \{y \in S : d(y, x_j) \le d(y, x_\ell) \text{ for all } x_\ell \in X_{J^A} \text{ and} \\ &\qquad\qquad d(y, x_j) \le d(y, x_\ell) + \beta \text{ for all } x_\ell \in X_{J^D}\} \\ v^D_{J^A,J^D}(x_j) &:= \{y \in S : d(y, x_j) \le d(y, x_\ell) - \beta \text{ for all } x_\ell \in X_{J^A} \text{ and} \\ &\qquad\qquad d(y, x_j) \le d(y, x_\ell) \text{ for all } x_\ell \in X_{J^D}\}. \end{aligned}$$

For $i \in N_n$, the payoff of player i is $u_i : X^n_K \to \mathbb{R}$, defined as follows:

$$u_i(\boldsymbol{a}^A, \boldsymbol{a}^D) = \begin{cases} \dfrac{1}{\operatorname{card}\{h : a^A_h = a^A_i\}} \displaystyle\sum_{J^A,J^D \subset K} \lambda(v^A_{J^A,J^D}(a^A_i)) \mathbb{1}((\boldsymbol{a}^A, \boldsymbol{a}^D) \approx X_{J^A,J^D}), & \text{if } i \in N^A_n, \\ \dfrac{1}{\operatorname{card}\{h : a^D_h = a^D_i\}} \displaystyle\sum_{J^A,J^D \subset K} \lambda(v^D_{J^A,J^D}(a^D_i)) \mathbb{1}((\boldsymbol{a}^A, \boldsymbol{a}^D) \approx X_{J^A,J^D}), & \text{if } i \in N^D_n. \end{cases}$$

We call $\mathscr{D}_n := \langle S, \lambda, N^A_n, N^D_n, X_K, \beta, (u_i)\rangle$ a *Hotelling game with differentiated players.*

Note that, in any pure strategy profile of the game $\mathscr{D}_n$, a D-player gets a strictly positive payoff only if she chooses a location that is not chosen by any advantaged players.

The next example shows how substantially different the equilibria of a game $\mathscr{G}_n$ and of a game $\mathscr{D}_n$ can be.

Example 5.1. Let $S = [0, 1]$ with λ the Lebesgue measure on $[0, 1]$ and $X_K = \{0, 1\}$. The game $\mathscr{G}_2$ admits pure equilibria. Actually any pure or mixed profile is an equilibrium and gives the same payoff $1/2$ to both players.

Consider now the game $\mathscr{D}_2$ with one advantaged and one disadvantaged players. In the unique equilibrium of $\mathscr{D}_2$ both players randomize with probability $1/2$ over the two possible locations.

Indeed, in $\mathscr{D}_2$ there cannot be a pure equilibrium in which both players choose the same location since the disadvantaged player would get 0 and hence would strictly increase her payoff by deviating. Similarly, there cannot be a pure equilibrium in which players choose different locations, since the advantaged player would have an incentive to deviate to the location chosen by the disadvantaged player. Therefore, any equilibrium must be mixed. A simple computation proves that uniform randomization is the unique strategy profile that constitutes an equilibrium.

We first turn our attention to pure equilibria. The following theorem is the analogue of Theorem 3.4 in the context of differentiated players.

Theorem 5.2. *Consider a sequence of games $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$. There exists $\bar{n}$ such that for all $n^A \geq \bar{n}$ the game $\mathscr{D}_n$ admits a pure equilibrium $\boldsymbol{a}^*$. Moreover, for all $n^A \geq \bar{n}$, any pure equilibrium satisfies*

$$\frac{n_j^A(\boldsymbol{a}^*)}{n_\ell^A(\boldsymbol{a}^*)+1} \leq \frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))} \leq \frac{n_j^A(\boldsymbol{a}^*)+1}{n_\ell^A(\boldsymbol{a}^*)}. \tag{5.1}$$

We now examine symmetric mixed equilibria in this model with differentiated candidates. Given a game $\mathscr{D}_n$, an equilibrium profile $(\boldsymbol{\gamma}^{A,n}, \boldsymbol{\gamma}^{D,n})$ is called (A, D)-symmetric if

$$\boldsymbol{\gamma}^{A,n} = (\gamma^{A,n}, \dots, \gamma^{A,n}), \tag{5.2}$$

$$\boldsymbol{\gamma}^{D,n} = (\gamma^{D,n}, \dots, \gamma^{D,n}). \tag{5.3}$$

Theorem 5.3. *For every $n \in \mathbb{N}$ the game $\mathscr{D}_n$ admits an (A, D)-symmetric equilibrium $(\boldsymbol{\gamma}^{A,n}, \boldsymbol{\gamma}^{D,n})$ such that*

$$\lim_{n^A\to\infty} \gamma^{A,n}(x_j) = \frac{\lambda(v_{K,J^D}^A(x_j))}{\lambda(S)} = \frac{\lambda(v_K(x_j))}{\lambda(S)} \quad \textit{for all } x_j \in S, \quad \textit{for all } J^D \subset K. \tag{5.4}$$

Moreover, in this equilibrium,

$$\lim_{n^A\to\infty} \sum_{i\in N^D} U_i^D(\boldsymbol{\gamma}^{A,n}, \boldsymbol{\gamma}^{D,n}) = 0. \tag{5.5}$$

Theorem 5.3 shows that, as the number n^A of advantaged players grows, they behave as if the disadvantaged players did not exist, so they play the same mixed strategies as in the game $\mathscr{G}_{n^A}$. The disadvantaged players in turn get a zero payoff whatever they do.

A Proofs

Section 3

Proof of Proposition 3.3. Pick any pair of locations $x_j, x_h \in X_K$ and consider the strategy profile $a_1 = x_j$ and $a_i = x_h$ for $i \neq 1$. Then, given assumption (2.1), for $i \neq 1$ and n sufficiently large, we have

$$u_1(\boldsymbol{a}) = \lambda\left(v_{\{j,h\}}(x_j)\right) \geq \frac{1}{n-1}\lambda\left(v_{\{j,h\}}(x_h)\right) = u_i(\boldsymbol{a}),$$

which shows that x_j is not weakly dominated. Given that the pair x_j, x_h was arbitrarily chosen, we have the result. $\square$

For any profile $\boldsymbol{a} \in X_K^n$ define

$$n_j(\boldsymbol{a}) := \operatorname{card}\{i \in N_n : a_i = x_j\}$$

the number of players who choose action x_j.

Lemma A.1. *Consider a sequence of games $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$. There exists $\bar{n}$ such that for all $n \geq \bar{n}$, if $\boldsymbol{a}^*$ is an equilibrium of $\mathscr{G}_n$, then*

$$n_j(\boldsymbol{a}^*) > 0 \quad \textit{for all } x_j \in X_K. \tag{A.1}$$

Proof. Assume by contradiction that for all $n \in \mathbb{N}$, if the game $\mathscr{G}_n$ has a pure equilibrium $\boldsymbol{a}^*$, then there exists a location $x_j \in X_K$ such that $n_j(\boldsymbol{a}^*) = 0$. We know that

$$\sum_{i\in N_n} u_i(\boldsymbol{a}^*) = \lambda(S) < \infty.$$

Therefore there exists $i \in N_n$ such that

$$u_i(\boldsymbol{a}^*) \leq \frac{\lambda(S)}{n}.$$

If this player deviated to $a_i = x_j$, she would achieve the payoff

$$u_i(a_i, \boldsymbol{a}^*_{-i}) \geq \lambda(v_K(x_j)) \geq \frac{\lambda(S)}{n},$$

for n large enough. This contradicts the assumption that $\boldsymbol{a}^*$ is a Nash equilibrium. □

Lemma A.2. *A strategy profile $\boldsymbol{a}^*$ is an equilibrium of the game $\mathscr{G}_n$ such that $n_j(\boldsymbol{a}^*) > 0$ for all $j \in K$ if and only if, for every $j, \ell \in K$,*

$$\frac{n_j(\boldsymbol{a}^*)}{n_\ell(\boldsymbol{a}^*) + 1} \leq \frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))} \leq \frac{n_j(\boldsymbol{a}^*) + 1}{n_\ell(\boldsymbol{a}^*)}. \tag{A.2}$$

Proof of Lemma A.2. Let $\boldsymbol{a}^*$ be an equilibrium of $\mathscr{G}_n$ and let $a_i^* = x_\ell$. Assume, by contradiction, that

$$\frac{\lambda(v_K(x_j))}{n_j(\boldsymbol{a}^*) + 1} > \frac{\lambda(v_K(x_\ell))}{n_\ell(\boldsymbol{a}^*)}.$$

Then player i could profitably deviate from x_ℓ to x_j. Therefore, for every $j, \ell \in K$ we have

$$\frac{\lambda(v_K(x_j))}{n_j(\boldsymbol{a}^*) + 1} \leq \frac{\lambda(v_K(x_\ell))}{n_\ell(\boldsymbol{a}^*)} \quad \text{and} \quad \frac{\lambda(v_K(x_\ell))}{n_\ell(\boldsymbol{a}^*) + 1} \leq \frac{\lambda(v_K(x_j))}{n_j(\boldsymbol{a}^*)} \tag{A.3}$$

and, applying Lemma A.1, (A.2) follows.

To prove the converse implication, assume that (A.2) holds. Equivalently, (A.3) holds for every $j, \ell \in K$. As a consequence, no player can profitably deviate from x_j to x_ℓ or vice versa, for every $j, \ell \in K$. Hence $\boldsymbol{a}^*$ is an equilibrium. □

Proof of Theorem 3.4. Given Lemmata A.1 and A.2, all we have to prove is that there exists $\bar{n}$ such that for each $n \geq \bar{n}$ there exist integers $n_1, \ldots, n_k$ such that

$$\sum_{j \in K} n_j = n, \quad \text{and, for all } j \in K,\ n_j > 0 \quad \text{and}$$

$$\frac{n_j}{n_\ell + 1} \leq \frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))} \leq \frac{n_j + 1}{n_\ell}. \tag{A.4}$$

Define

$$\beta_j = \frac{\lambda(v_k(x_j))}{\sum_{\ell \in K} \lambda(v_k(x_\ell))}$$

and

$$\bar{n} := \min\{n \mid n\beta_\ell - 1 > 0 \text{ for all } \ell \in K\}. \tag{A.5}$$

If we sum the inequalities in (A.4) over $j \in K$, we get

$$\frac{n}{n_\ell + 1} \leq \frac{1}{\beta_\ell} \leq \frac{n_\ell + 1}{n},$$

which, after some simple algebra, becomes

$$n\beta_\ell - 1 \leq n_\ell \leq (n + k)\beta_\ell, \tag{A.6}$$

where k is the cardinality of K. Notice that $n\beta_\ell - 1 > 0$, by (A.5), and the set of admissible values for n_ℓ is the set of all integers in an interval of length $k\beta_\ell + 1$. Without any loss of generality, assume $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_k$.

We describe a simple algorithm that provides the desired $n_1, \ldots, n_k$. For $\ell \in K$ take

$$\tilde{n}_\ell = \lfloor n\beta_\ell \rfloor.$$

If

$$\sum_{j \in K} \tilde{n}_j = n, \tag{A.7}$$

then the vector $\tilde{n}_1, \ldots, \tilde{n}_k$ is the desired vector. If not, increase $\tilde{n}_k$ by 1 and check whether (A.7) holds. If not, and if $\tilde{n}_{k-1} + 1 \leq (n + k)\beta_{k-1}$, then increase $\tilde{n}_{k-1}$ by one, and check whether (A.7) holds. Continue until either (A.7) holds or you reach an index h such that $\tilde{n}_h + 1 > (n+k)\beta_h$. If this happens go back to $\tilde{n}_k$ and increase it by 1. The procedure ends in finite time. Once we have a positive vector $(\tilde{n}_1, \ldots, \tilde{n}_k)$ that for all $\ell \in K$ satisfies (A.6) and hence (A.4), all we have to do is to devise a strategy profile $\boldsymbol{a}^*$ such that

$$n_j(\boldsymbol{a}^*) = \tilde{n}_j. \qquad \square$$

The proof of Theorem 3.5 requires some preliminary results.

Lemma A.3. *Consider a sequence of games $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$. There exists $\bar{n}$ such that for all $n \geq \bar{n}$, if $\boldsymbol{\gamma}^{(n)}$ is a symmetric equilibrium of $\mathscr{G}_n$, then $\boldsymbol{\gamma}^{(n)}$ is completely mixed, i.e.,*

$$\gamma^{(n)}(x_j) > 0 \quad \textit{for all } x_j \in X_K.$$

Proof. Assume by contradiction that for every $n \in \mathbb{N}$ there exists some $x_j \in X_K$ and a symmetric equilibrium $\boldsymbol{\gamma}^{(n)}$ of $\mathscr{G}_n$ such that $\gamma^{(n)}(x_j) = 0$. Given that $\lambda(S) < \infty$, we have that for all $i \in N_n$

$$U_i(\boldsymbol{\gamma}^{(n)}) = \frac{\lambda(S)}{n}.$$

If player i deviates and plays the pure action $a_i = x_j$, then she obtains a payoff

$$U_i(a_i, \boldsymbol{\gamma}_{-i}^{(n)}) \geq \lambda(v_K(x_j)) \geq \frac{\lambda(S)}{n},$$

for n large enough. This contradicts the assumption that $\boldsymbol{\gamma}^{(n)}$ is an equilibrium. $\square$

Lemma A.4. *Let $(Y_1, \ldots, Y_k)$ be a random vector distributed according to a multinomial distribution with parameters $(n-1; \gamma_1^{(n)}, \ldots, \gamma_k^{(n)})$, with $\delta < \gamma_j^{(n)} < 1-\delta$, for some $0 < \delta < 1$ and for all $j \in K$. Then*

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[\dfrac{1}{Y_j+1}\displaystyle\sum_{J\subset K} \lambda(v_J(x_j))\mathbb{1}(Y_h = 0 \textit{ for } h \notin J)\right]}{\mathbb{E}\left[\dfrac{1}{Y_\ell+1}\displaystyle\sum_{J\subset K} \lambda(v_J(x_\ell))\mathbb{1}(Y_h = 0 \textit{ for } h \notin J)\right]} = 1, \quad \textit{for all } j, \ell \in K \tag{A.8}$$

iff

$$\lim_{n\to\infty} \gamma_j^{(n)} = \gamma(x_j) = \frac{\lambda(v_K(x_j))}{\lambda(S)} \quad \textit{for all } j \in K. \tag{A.9}$$

Proof. Given $j \in K$, consider all $J \subset K$ such that $j \in J$ and the family $\mathscr{V}_j$ of all corresponding Voronoi tessellations $V(X_J)$. Call $\widetilde{V}_j$ the finest partition of S generated by $\mathscr{V}_j$, that is, the set of all possible intersections of cells $v_J(x_j) \in V(X_J)$ for $V(X_J) \in \mathscr{V}_j$. It is clear that $v_K(x_j) \in \widetilde{V}_j$.

For $A \in \widetilde{V}_j$, call $\widetilde{V}_j(A)$ the class of all cells in $\widetilde{V}_j$ whose intersection with A is nonempty. Then

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{Y_j+1}\sum_{J\subset K}\lambda(v_J(x_j))\mathbb{1}(Y_h = 0 \text{ for } h \notin J)\right] &= \mathbb{E}\left[\frac{\lambda(v_K(x_j))}{Y_j+1}\right] \\
&\quad + \mathbb{E}\left[\frac{1}{Y_j+1}\sum_{A\in\widetilde{V}_j}\lambda(A)\mathbb{1}\left(Y_h = 0 \text{ if } v_K(x_j)\cap A \neq \varnothing\right)\right] \\
&\leq \mathbb{E}\left[\frac{\lambda(v_K(x_j))}{Y_j+1}\right] \\
&\quad + \sum_{A\in\widetilde{V}_j}\lambda(A)\mathbb{P}\left(Y_h = 0 \text{ if } v_K(x_j)\cap A \neq \varnothing\right) \\
&= \mathbb{E}\left[\frac{\lambda(v_K(x_j))}{Y_j+1}\right] + o(1/n) \quad \text{for } n \to \infty,
\end{aligned}$$

since $\mathbb{P}(Y_i = 0) = (1-\gamma_i^{(n)})^n = o(1/n)$ for $n \to \infty$. Therefore

$$\begin{aligned}
\lim_{n\to\infty}\frac{\mathbb{E}\left[\frac{1}{Y_j+1}\sum_{J\subset K}\lambda(v_J(x_j))\mathbb{1}(Y_h = 0 \text{ for } h \notin J)\right]}{\mathbb{E}\left[\frac{1}{Y_\ell+1}\sum_{J\subset K}\lambda(v_J(x_\ell))\mathbb{1}(Y_h = 0 \text{ for } h \notin J)\right]} &= \lim_{n\to\infty}\frac{\mathbb{E}\left[\frac{\lambda(v_K(x_j))}{Y_j+1}\right]}{\mathbb{E}\left[\frac{\lambda(v_K(x_\ell))}{Y_\ell+1}\right]} \\
&= \lim_{n\to\infty}\frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))}\frac{\gamma_\ell^{(n)}}{\gamma_j^{(n)}} \qquad \text{(A.10)} \\
&= \frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))}\frac{\gamma(x_\ell)}{\gamma(x_j)}.
\end{aligned}$$

Given that $\sum_{j=1}^{k}\gamma(x_j) = 1$, (A.10) holds if and only if (A.9) does. ☐

Proof of Theorem 3.5. The game $\mathscr{G}_n$ is finite and symmetric, so it admits a symmetric mixed Nash equilibrium $\boldsymbol{\gamma}^{(n)} = (\gamma^{(n)}, \dots, \gamma^{(n)})$. Then, given Lemma A.3, for all

$j, \ell \in K$,

$$U_i(x_j, \boldsymbol{\gamma}_{-i}^{(n)}) = U_i(x_\ell, \boldsymbol{\gamma}_{-i}^{(n)}). \tag{A.11}$$

Using (2.2) we obtain

$$\begin{aligned} U_i(x_j, \boldsymbol{\gamma}_{-i}^{(n)}) &= \sum_{a_1 \in X_K} \cdots \sum_{a_n \in X_K} u_i(a_1, \dots, a_{i-1}, x_j, a_{i+1}, \dots, a_n) \\ &\qquad \gamma^{(n)}(x_1)^{n_1(\boldsymbol{a}_{-i})} \dots \gamma^{(n)}(x_j)^{n_j(\boldsymbol{a}_{-i})+1} \dots \gamma^{(n)}(x_k)^{n_k(\boldsymbol{a}_{-i})} \\ &= \mathbb{E}\left[\frac{1}{Y_j+1} \sum_{J \subset K} \lambda(v_J(x_j)) \mathbb{1}(Y_h = 0 \text{ for } h \notin J)\right], \end{aligned}$$

where $(Y_1, \dots, Y_k)$ has a multinomial distribution with parameters $(n-1; \gamma^{(n)}(x_1), \dots, \gamma^{(n)}(x_k))$. Notice that $\boldsymbol{a} \approx X_J$ is equivalent to $Y_h = 0$ for all $h \notin J$.

Therefore (A.11) holds if and only if

$$\begin{aligned} &\mathbb{E}\left[\frac{1}{Y_j+1} \sum_{J \subset K} \lambda(v_J(x_j)) \mathbb{1}(Y_h = 0 \text{ for } h \notin J)\right] \\ &\qquad\qquad = \mathbb{E}\left[\frac{1}{Y_\ell+1} \sum_{J \subset K} \lambda(v_J(x_\ell)) \mathbb{1}(Y_h = 0 \text{ for } h \notin J)\right], \end{aligned}$$

which implies (A.8). Lemma A.4 provides the result. □

Section 4

The next two lemmata are similar to Lemmata A.3 and A.4, respectively.

Lemma A.5. *Consider a sequence of games $\{\mathscr{P}_n\}_{n \in \mathbb{N}}$. There exists $\bar{n}$ such that for all $n \geq \bar{n}$, if $\boldsymbol{\gamma}^{(n)}$ is a symmetric equilibrium of $\mathscr{P}_n$, then $\boldsymbol{\gamma}^{(n)}$ is completely mixed, i.e.,*

$$\gamma^{(n)}(x_j) > 0 \quad \textit{for all } x_j \in X_K.$$

Proof. Assume by contradiction that for every $n \in \mathbb{N}$ there exists some $x_j \in X_K$ and a symmetric equilibrium $\boldsymbol{\gamma}^{(n)}$ of $\mathscr{P}_n$ such that $\gamma^{(n)}(x_j) = 0$. Given that $\lambda(S) < \infty$, we have that for each player i

$$U_i(\boldsymbol{\gamma}^{(n)}) = \mathbb{E}\left[\frac{\lambda(S)}{\Xi_n}\right],$$

where Ξ_n has a Poisson distribution with parameter n. If player i deviates and plays the pure action $a_i = x_j$, then she obtains a payoff

$$U_i(a_i, \boldsymbol{\gamma}_{-i}^{(n)}) \geq \lambda(v_K(x_j)) \geq \mathbb{E}\left[\frac{\lambda(S)}{\Xi_n}\right],$$

for n large enough. This contradicts the assumption that $\boldsymbol{\gamma}^{(n)}$ is an equilibrium. $\square$

Lemma A.6. *Let $(\Xi_1, \ldots, \Xi_k)$ be a random vector of independent random variables where Ξ_j has a Poisson distribution with parameter $n\gamma_j^{(n)}$, with $\delta < \gamma_j^{(n)} < 1 - \delta$, for some $0 < \delta < 1$ and for all $j \in K$. Then*

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[\dfrac{1}{\Xi_j + 1}\displaystyle\sum_{J\subset K}\lambda(v_J(x_j))\mathbb{1}(\Xi_h = 0 \text{ for } h \notin J)\right]}{\mathbb{E}\left[\dfrac{1}{\Xi_\ell + 1}\displaystyle\sum_{J\subset K}\lambda(v_J(x_\ell))\mathbb{1}(\Xi_h = 0 \text{ for } h \notin J)\right]} = 1, \quad \text{for all } j, \ell \in K \tag{A.12}$$

iff

$$\lim_{n\to\infty} \gamma_j^{(n)} = \gamma(x_j) = \frac{\lambda(v_K(x_j))}{\lambda(S)} \quad \text{for all } j \in K. \tag{A.13}$$

Proof. Given $j \in K$, consider all $J \subset K$ such that $j \in J$ and the family $\mathscr{V}_j$ of all corresponding Voronoi tessellations $V(X_J)$. Call $\widetilde{V}_j$ the finest partition of S generated by $\mathscr{V}_j$, that is, the set of all possible intersections of cells $v_J(x_j) \in V(X_J)$ for $V(X_J) \in \mathscr{V}_j$. It is clear that $v_K(x_j) \in \widetilde{V}_j$.

For $A \in \widetilde{V}_j$, call $\widetilde{V}_j(A)$ the class of all cells in $\widetilde{V}_j$ whose intersection with A is nonempty. Then

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{\Xi_j+1}\sum_{J\subset K}\lambda(v_J(x_j))\mathbb{1}(\Xi_h=0 \text{ for } h\notin J)\right] &= \mathbb{E}\left[\frac{\lambda(v_K(x_j))}{\Xi_j+1}\right] \\
&\quad + \mathbb{E}\left[\frac{1}{\Xi_j+1}\sum_{A\in\widetilde{V}_j}\lambda(A)\mathbb{1}\left(\Xi_h=0 \text{ if } v_K(x_j)\cap A\neq\varnothing\right)\right] \\
&\leq \mathbb{E}\left[\frac{\lambda(v_K(x_j))}{\Xi_j+1}\right] \\
&\quad + \sum_{A\in\widetilde{V}_j}\lambda(A)\mathbb{P}\left(\Xi_h=0 \text{ if } v_K(x_j)\cap A\neq\varnothing\right) \\
&= \mathbb{E}\left[\frac{\lambda(v_K(x_j))}{\Xi_j+1}\right] + o(1/n) \quad \text{for } n\to\infty,
\end{aligned}$$

since $\mathbb{P}(\Xi_i=0)=\mathrm{e}^{-n}=o(1/n)$ for $n\to\infty$. Therefore

$$\begin{aligned}
\lim_{n\to\infty}\frac{\mathbb{E}\left[\frac{1}{\Xi_j+1}\sum_{J\subset K}\lambda(v_J(x_j))\mathbb{1}(\Xi_h=0 \text{ for } h\notin J)\right]}{\mathbb{E}\left[\frac{1}{\Xi_\ell+1}\sum_{J\subset K}\lambda(v_J(x_\ell))\mathbb{1}(\Xi_h=0 \text{ for } h\notin J)\right]} &= \lim_{n\to\infty}\frac{\mathbb{E}\left[\frac{\lambda(v_K(x_j))}{\Xi_j+1}\right]}{\mathbb{E}\left[\frac{\lambda(v_K(x_\ell))}{\Xi_\ell+1}\right]} \\
&= \lim_{n\to\infty}\frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))}\frac{\gamma_\ell^{(n)}}{\gamma_j^{(n)}} \qquad \text{(A.14)} \\
&= \frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))}\frac{\gamma(x_\ell)}{\gamma(x_j)}
\end{aligned}$$

Given that $\sum_{j=1}^{k}\gamma(x_j)=1$, (A.14) holds if and only if (A.13) does. $\square$

Proof of Theorem 4.1. Since the number of types and actions is finite, Myerson (1998, Theorem 3) implies that the Poisson game $\mathscr{P}_n$ admits a symmetric equilibrium $\boldsymbol{\gamma}^{(n)}$. Given Lemma A.5, for all $j,\ell\in K$,

$$U_i(x_j,\boldsymbol{\gamma}_{-i}^{(n)}) = U_i(x_\ell,\boldsymbol{\gamma}_{-i}^{(n)}). \qquad \text{(A.15)}$$

For $j\in K$ call $n_j(\boldsymbol{a},\xi)$ the number of players who choose x_j under strategy $\boldsymbol{a}$ when the total number of players in the game is ξ. Using (2.2) we obtain

$$U_i(x_j, \boldsymbol{\gamma}_{-i}^{(n)}) = \sum_{\xi=1}^{\infty} \Bigg[\sum_{a_1 \in X_K} \cdots \sum_{a_\xi \in X_K} u_i(a_1, \dots, a_{i-1}, x_j, a_{i+1}, \dots, a_\xi)$$

$$\gamma^{(n)}(x_1)^{n_1(\boldsymbol{a}_{-i},\xi)} \dots \gamma^{(n)}(x_j)^{n_j(\boldsymbol{a}_{-i},\xi)+1} \dots \gamma^{(n)}(x_k)^{n_k(\boldsymbol{a}_{-i},\xi)} \Bigg] \frac{\mathrm{e}^{-n} n^\xi}{\xi!}$$

$$= \mathbb{E}\left[\frac{1}{\Xi_j + 1} \sum_{J \subset K} \lambda(v_J(x_j)) \mathbb{1}(\Xi_h = 0 \text{ for } h \notin J) \right],$$

where $(\Xi_1, \dots, \Xi_k)$ are independent random variables such that Ξ_j has a Poisson distribution with parameter $n\gamma^{(n)}(x_j)$. Notice that $\boldsymbol{a} \approx X_J$ is equivalent to $\Xi_h = 0$ for all $h \notin J$.

Therefore (A.15) holds if and only if

$$\mathbb{E}\left[\frac{1}{\Xi_j + 1} \sum_{J \subset K} \lambda(v_J(x_j)) \mathbb{1}(\Xi_h = 0 \text{ for } h \notin J) \right]$$
$$= \mathbb{E}\left[\frac{1}{\Xi_\ell + 1} \sum_{J \subset K} \lambda(v_J(x_\ell)) \mathbb{1}(\Xi_h = 0 \text{ for } h \notin J) \right],$$

which implies (A.12). Lemma A.6 provides the result. □

Section 5

Lemma A.7. *Consider a sequence of games $\{\mathscr{D}_n\}_{n \in \mathbb{N}}$. There exists $\bar{n}$ such that for all $n^A \geq \bar{n}$, if $\boldsymbol{a}^*$ is an equilibrium of $\mathscr{D}_n$, then*

$$n_j^A(\boldsymbol{a}^*) > 0 \quad \textit{for all } x_j \in X_K. \tag{A.16}$$

Proof. Assume by contradiction that for all $n \in \mathbb{N}$, if the game $\mathscr{D}_n$ has a pure equilibrium $\boldsymbol{a}^*$, then there exists a location $x_j \in X_K$ such that $n_j^A(\boldsymbol{a}^*) = 0$. We know that

$$\sum_{i \in N_n} u_i(\boldsymbol{a}^*) = \lambda(S) < \infty.$$

Therefore there exists $i \in N_n^A$ such that

$$u_i(\boldsymbol{a}^*) \le \frac{\lambda(S)}{n^A + n^B}.$$

If this player deviated to $a_i = x_j$, she would achieve the payoff

$$u_i(a_i, \boldsymbol{a}^*_{-i}) \ge \lambda(v_K(x_j)) \ge \frac{\lambda(S)}{n^A + n^B},$$

for n^A large enough. This contradicts the assumption that $\boldsymbol{a}^*$ is a Nash equilibrium.

□

Proof of Theorem 5.2. Given Lemma A.7, *mutatis mutandis* the proof is similar to the proof of Theorem 3.4, and is therefore omitted. □

Lemma A.8. *Consider a sequence of games $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$. There exists $\bar{n}^A$ such that for all $n^A \ge \bar{n}^A$, if $(\boldsymbol{\gamma}^{A,n}, \boldsymbol{\gamma}^{D,n})$ is an (A, D)-symmetric equilibrium of $\mathscr{D}_n$, then $\gamma^{A,n}$ is completely mixed, i.e.,*

$$\gamma^{A,n}(x_j) > 0 \quad \textit{for all } x_j \in X_K.$$

Proof. Assume by contradiction that for every $n \in \mathbb{N}$ there exists some $x_j \in X_K$ and an (A, D)-symmetric equilibrium $(\boldsymbol{\gamma}^{A,n}, \boldsymbol{\gamma}^{D,n})$ of $\mathscr{D}_n$, such that $\gamma^{A,n}(x_j) = 0$. Given that $\lambda(S) < \infty$, we have that for $i \in N_n^A$

$$U_i^A(\boldsymbol{\gamma}^{A,n}, \boldsymbol{\gamma}^{D,n}) \le \frac{\lambda(S)}{n^A}.$$

If player $i \in N_n^A$ deviates and plays the pure action $a_i = x_j$, then she obtains a payoff

$$U_i^A(a_i, \boldsymbol{\gamma}^{A,n}_{-i}, \boldsymbol{\gamma}^{D,n}) \ge \lambda(v_K(x_j)) \ge \frac{\lambda(S)}{n^A},$$

for n^A large enough. Indeed, note that even if some D-players choose x_j in $\gamma^{D,n}$, the A player attracts all the consumers from x_j. Therefore $(\boldsymbol{\gamma}^{A,n}, \boldsymbol{\gamma}^{D,n})$ is not an equilibrium for n^A large enough. □

Lemma A.9. *Let $(Y_1, \dots, Y_k)$ be a random vector distributed according to a multinomial distribution with parameters $(n; \gamma_1^{(n)}, \dots, \gamma_k^{(n)})$, with $\delta < \gamma_j^{(n)} < 1 - \delta$, for some $0 < \delta < 1$ and for all $j \in K$. Then*

$$\lim_{n \to \infty} \mathbb{P}(Y_j = 0) = 0 \quad \textit{for all } j \in K.$$

Proof. The result is obvious, since

$$\mathbb{P}(Y_j = 0) = (1 - \gamma_j^{(n)})^n \leq (1 - \delta)^n \to 0.$$

□

Proof of Theorem 5.3. Whenever a location x_j is occupied by an advantaged player, any disadvantaged player choosing x_j gets a payoff equal to zero. Therefore (5.5) is an immediate consequence of Lemmata A.8 and A.9. Moreover, asymptotically, the actions of disadvantaged players do not affect the payoff of advantaged players. Therefore an application of Lemma A.4 with n^A replacing n provides (5.4). □

References

Aoyagi, M. and Okabe, A. (1993) Spatial competition of firms in a two-dimensional bounded market. *Reg. Sci. Urban Econ.* **23**, 259–289.

Aragones, E. and Palfrey, T. R. (2002) Mixed equilibrium in a Downsian model with a favored candidate. *J. Econom. Theory* **103**, 131–161. Political science.

Aragonès, E. and Xefteris, D. (2012) Candidate quality in a Downsian model with a continuous policy space. *Games Econom. Behav.* **75**, 464–480.

Downs, A. (1957) *An Economic Theory of Democracy.* Harper and Row, New York.

Dürr, C. and Thang, N. K. (2007) Nash equilibria in Voronoi games on graphs. In *European Symposium on Algorithms.*

Eaton, B. C. and Lipsey, R. G. (1975) The principle of minimum differentiation reconsidered: some new developments in the theory of spatial competition. *Rev. Econ. Stud.* **42**, 27–49.

Feldmann, R., Mavronicolas, M., and Monien, B. (2009) Nash equilibria for Voronoi games on transitive graphs. In Leonardi, S. (ed.), *Internet and Network Economics*, volume 5929 of *Lecture Notes in Computer Science*, 280–291. Springer Berlin Heidelberg.

Fournier, G. and Scarsini, M. (2014) Hotelling games on networks: efficiency of equilibria. SSRN 2423345.

Gouret, F., Hollard, G., and Rossignol, S. (2011) An empirical analysis of valence in electoral competition. *Soc. Choice Welf.* **37**, 309–340.

Gur, Y., Saban, D., and Stier-Moses, N. E. (2014) The competitive facility location problem in a duopoly. Mimeo.

Heijnen, P. and Soetevent, A. R. (2014) Price competition on graphs. Technical Report TI 2014-131/VII, Tinbergen Institute.

Hörner, J. and Jamison, J. (2012) Hotelling's spatial model with finitely many consumers. Mimeo.

Hotelling, H. (1929) Stability in competition. *Econ. J.* **39**, 41–57.

Laster, D., Bennet, P., and Geoum, I. (1999) Rational bias in macroeconomic forecasts. *Quart. Journal of Econ.* **45**, 145–186.

Lederer, P. J. and Hurter, Jr., A. P. (1986) Competition of firms: discriminatory pricing and location. *Econometrica* **54**, 623–640.

Mavronicolas, M., Monien, B., Papadopoulou, V. G., and Schoppmann, F. (2008) Voronoi games on cycle graphs. In *Mathematical Foundations of Computer Science 2008*, volume 5162 of *Lecture Notes in Comput. Science*, 503–514. Springer, Berlin.

Myerson, R. B. (1998) Population uncertainty and Poisson games. *Internat. J. Game Theory* **27**, 375–392.

Myerson, R. B. (2000) Large Poisson games. *J. Econom. Theory* **94**, 7–45.

Osborne, M. J. and Pitchik, C. (1986) The nature of equilibrium in a location model. *Internat. Econom. Rev.* **27**, 223–237.

Ottaviani, M. and Sorensen, P. N. (2006) The strategy of professional forecasting. *J. Financ. Econ.* **81**, 441–466.

Pálvölgyi, D. (2011) Hotelling on graphs. Mimeo.

Suzuki, J. (2013) Land use regulation as a barrier to entry: evidence from the Texas lodging industry. *Int. Econ. Rev.* **54**, 495–523.

Tabuchi, T. (1994) Two-stage two-dimensional spatial competition between two firms. *Reg. Sci. Urban Econ.* **24**, 207–227.