Revealed Preferences: A Topological Approach

Gyula Magyarkuti

Department of Mathematics, Corvinus University of Budapest, P.O.Box 489, Budapest, Hungary 1828

Abstract

The purpose of this paper is to study the relationship between the axiomatic foundations of revealed preference theory and the continuity properties of choice. The main result of this paper shows that the continuity of a set-to-point choice function is equivalent to the weak axiom of revealed preference and openness of the strict revealed relation, provided that the collection of budget sets is endowed with a topology used widely by economists.

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1. Introduction

The purpose of this paper is to study the relationship between axiomatic foundations of revealed preference theory and the continuity properties of choice. Several papers have studied the replacement of some preference assumptions with topological conditions. We only mention the important and elegant papers by Sonnenschein (1965) and Schmeidler (1971). The alternative approach to the theory of consumer behavior is to study choice correspondences or functions. The *rationalization problem* is to assure the existence of a preference which generates the choice correspondence in some sense. The assumptions used in solving this problem are called revealed preference axioms. The connections among these assumptions have been thoroughly investigated, we only refer to papers by Richter (1966) and Richter (1971), Suzumura (1976) and Clark (1985) based on the works of Samuelson (1947) and Arrow (1959).

These works do not use topological methods, however. Our intention in the present paper is to point out that the topological approach to revealed preference theory can lead to new results on the rationalization problem. Our investigations are carried out in two frameworks. The "special framework" means that the choice correspondence is single-valued, that is, it is a choice *function*. In the "general framework", there is no restriction on the chosen sets.

Email address: magyarkuti@member.ams.org (Gyula Magyarkuti)
URL: http://www.uni-corvinus.hu/magyarkuti (Gyula Magyarkuti)

(1) In the "special framework", we verify that the so–called Vietoris continuity of the choice function is equivalent to the weak axiom (WARP) and openness of the revealed preference. The technique that we exploit for the proof was originally developed in E. Michael's famous paper – Michael (1951) –, which uses the seminal paper of Eilenberg (1941).

(2) In the "general framework", we provide a supplementary condition that can be assumed in addition to WARP, so that the choice correspondence is upper hemicontinuous.

2. Definitions and notation

$\mathcal{J}_n(X)$, $\mathcal{J}(X)$, $\mathcal{K}(X)$ and $\mathcal{P}(X)$ denote the collection of all subsets of the universal set X with at most n elements, the collection of all finite subset of X, the collection of all compact subset of X and the set of all subset of X, respectively. Let $A(R) = R \cap \left(R^{-1}\right)^c$ stand for the asymmetric part and $S(R) = R \cap R^{-1}$ denotes the symmetric part of a binary relation $R \subseteq X \times X$.

The triple $(X, \mathcal{B}, c)$ is called a *decision structure* where $\mathcal{B}$ is a subset of the power set of X and $c : \mathcal{B} \to \mathcal{P}(X)$ is a choice correspondence such that $\emptyset \neq c(B) \subseteq B$. If a binary relation $R \subseteq X \times X$ is given, denote by $c^R(B)$ the R-greatest elements of the set B, that is, $c^R(B) \doteq \{x \in B : (x, y) \in R \ \forall y \in B\}$. The choice correspondence c of a decision structure is said to be *transitive–rational* if there exists a binary, transitive relation R such that $c(B) = c^R(B)$ for any $B \in \mathcal{B}$.

The pioneering works by Samuelson (1947), Houthakker (1950), Uzawa (1956), and Arrow (1959) introduced two kinds of definition for the (directly) *revealed preference relation*. The first one is the *weak* $(x, y) \in R$ if and only if there exists $B \in \mathcal{B}$ for which $x \in c(B)$ and $y \in B$, while the second one is the *strict* $(x, y) \in P$ if and only if there exists $B \in \mathcal{B}$ for which $x \in c(B)$ and $y \in B \setminus c(B)$. The revealed preference axioms can be drawn up exploiting these two kinds of revealed relations. The most succinct formulation of the *weak axiom of revealed preference* (WARP) is the inclusion $R \subseteq \left(P^{-1}\right)^c$.

Let (X, τ) be a topological space, and for any open set $U \in \tau$,

$$U^+ \doteq \{B \in \mathcal{P}(X) : B \subseteq U\} \quad \text{and} \quad U^- \doteq \{B \in \mathcal{P}(X) : B \cap U \neq \emptyset\}.$$

The set $\{U^+ : U \in \tau\}$ forms the subbase of the *upper Vietoris topology* and $\{V^- : V \in \tau\}$ is the subbase of the *lower Vietoris topology* on the hyperspace of X. The *Vietoris* (or *finite*) *topology* is generated by the upper and lower Vietoris open sets. The elements of the filter base of this topology are sets of the form $\langle V_1, ..., V_n \rangle \doteq \{B \subseteq X : B \subseteq \cup_{i=1}^n V_i$ and $B \cap V_i \neq \emptyset\}$ where $V_i \in \tau$, and n is a positive integer. We refer to Hildenbrand (1974) and (Klein and Thompson, 1984) concerning basic results on Vietoris topology and applications to mathematical economics. Note that a set-to-set function is said to be *upper hemicontinuous* if it is continuous when its domain is supplied with the Vietoris topology and its range is endowed with the upper Vietoris topology.

Recall that relation R is *compatible with topology* τ if the upper and lower level sets R^x and R_x are closed sets and $A(R)^x$ and $A(R)_x$ are open sets. For the sake of simplicity, an asymmetric relation P is said to be *open* if the level sets P^x and P_x are open for any $x \in X$.

3. Results

The revealed preference axioms are drawn up about the nature of consistent behavior. The gist of the main theorem of this paper is that rationality of decision making can also be encapsulated by some continuity property of choice.

Theorem 1. *Suppose that (X, τ) is a connected separated topological space, $\mathcal{J}_n(X)$ carries the Vietoris topology and $c : \mathcal{J}_n(X) \to X$ is a choice function where $n \geq 3$. Then the continuity of c is equivalent to WARP together with openness of the strict revealed preference P.*

In this environment WARP means that the choice is transitive-rational. The theorem therefore characterizes transitive rationality combined with the openness of the strict revealed relation via the continuity of choice. This kind of continuity of choice at a finite set means, that if x_1 is chosen from the set $\{x_1, \ldots, x_n\}$ then the chosen element from the set $\{x'_1, \ldots, x'_n\}$ will be x'_1, if x'_i is close enough to x_i for $i = 1, \ldots, n$, respectively. This expresses an intrinsic property of a decision maker: a small pointwise change of available resources implies only small change of the chosen goods.

It may occur that a choice function is transitive–rational on two different connected components but the property of transitive–rationality fails with respect to the entire domain of choice. Of course, the continuity is independent of the components of the domain hence the connectedness is an unavoidable assumption.

The condition that the domain of choice must include all sets with at most n elements seems to be the most restrictive condition of the above theorem. This assumption is essentially the same as the one first appearing in Arrow (1959), when the domain includes all finite subsets of the set of available goods. Nevertheless, as Arrow pointed out, "requiring the choice functions to be defined for finite sets is thoroughly consistent with the intuitive arguments underlying revealed preference." As we will clarify, Theorem 1 remains valid if the domain of choice is substituted by a set of subsets $\mathcal{D}$, where $\mathcal{J}(X) \subseteq \mathcal{D} \subseteq \mathcal{K}(X)$ which is exactly the case motivated by Arrow.

4. Proofs

The sufficiency part of Theorem 1 is Proposition 3 which is based on the connectedness of X and the openness of some sets (Proposition 2). The necessity part is a consequence of Proposition 5, which can be regarded as a generalization of results stating the continuity of the demand correspondence. The propositions and their proofs are based on Michael's paper: Michael (1951).

4.1. Preliminaries

First we give a short summary of those statements and basic ideas of revealed preference theory that will be used to prove our theorems.

Richter (1971) gave the following characterization of transitive-rationalization. This result emphasizes the role of the weak revealed preference relation: a choice which is rationalizable with a transitive underlaying relation can also be rationalized with the transitive closure of the weak revealed relation.

(R.1) *The choice correspondence is transitive-rational if and only if* $c(B) = c^{t(R)}(B)$ *for any* $B \in \mathcal{B}$, *where* $t(R)$ *denotes the transitive closure of weak revealed relation* R.

Let us introduce the weak axioms following Richter (1971) and Suzumura (1976). The *weak congruence axiom* (WCA) says: $(x, y) \in R$ and $y \in c(B)$, $x \in B$ implies $x \in c(B)$. This means that if at one time x was chosen when y was available then in another time x has to be also chosen if y is chosen and x is available. WCA reflects that some kind of consistent behavior is required of the decision maker. The weak axiom of revealed preference (WARP) is: if x is weakly revealed preferred to y then y cannot be strictly revealed preferred to x, which is of course the same as we defined earlier. ($R \subseteq \left(P^{-1}\right)^c$, where R is the weak and P is the strict revealed relations.) Assuming that the chosen sets are singletons we get the theory which was originally formulated by Houthakker (1950). In this special case the condition that the strict revealed relation P be asymmetric is equivalent to WARP. In the general case the relationship of these axioms is also well known:

(R.2) *Let* $(X, \mathcal{B}, c)$ *an arbitrary decision structure,* R *the weak and* P *the strict revealed relations. Then the conditions WCA, WARP and the identity* $P = A(R)$ *are equivalent to each other. WARP implies the identity* $c = c^R$. *Conversely, the identity* $c = c^R$ *does not imply WARP.*

Let us consider now the strong axioms. The *strong congruence axiom* (SCA) says: $(x, y) \in t(R)$ and $y \in c(B)$, $x \in B$ implies $x \in c(B)$. *Hansson's axiom of revealed preference* (HARP) is: $t(R) \subseteq \left(P^{-1}\right)^c$. At first sight HARP is different from its traditional formulation that appeared in Suzumura (1976), but they are equivalent. The following theorem and (R.1) show that this stronger requirement of consistent behavior means transitive-rationality. The equivalence of SCA and transitive-rationality comes from Richter (1971), and the equivalence of SCA and HARP is due to Suzumura (1976). Originally, Richter (1966) showed that SCA is equivalent to rationality with a transitive and complete underlying relation.

(R.3) *Let* $(X, \mathcal{B}, c)$ *be an arbitrary decision structure and* R *the weak revealed relation. Then SCA, HARP and the equation* $c = c^{t(R)}$ *are equivalent.*

The theory becomes much simpler if we impose two restrictions on the domain of choice. We summarize the result of investigations initiated by Arrow (1959). This certifies that WARP coincides with the transitive-rationality of the choice in Theorem 1.

(R.4) *Let* $(X, \mathcal{B}, c)$ *be a decision structure. If* $\mathcal{J}_3(X) \subseteq \mathcal{B}$ *then WARP is equivalent to the identity* $c = c^R$ *together with the transitivity of weak revealed* *relation R. In this case each of the strong and each of the weak axioms coincides with the condition that c be transitive-rational.*

4.2. Proofs, remarks, counterexamples

Proposition 2. *Let (X,τ) be a separated topological space, $\mathcal{J}_n(X)$ supplied with the Vietoris topology, and $n \geq 2$ integer. Suppose that the choice function $c : \mathcal{J}_n(X) \to X$ is continuous and $x \neq y$, $x, y \in X$ are fixed.*
Then the following collections of sets $\{E \in \mathcal{J}_{n-2}(X) : c(E \cup \{x,y\}) \neq x\}$ and $\{E \in \mathcal{J}_{n-2}(X) : x \notin E, c(E \cup \{x,y\}) = x\}$ are open in the Vietoris topology on $\mathcal{J}_{n-2}(X)$.

Proof. Let $E \in \mathcal{J}_{n-2}(X)$ such that $c(E \cup \{x,y\}) = z \neq x$. Let V be a neighborhood of z that does not contain x. The continuity of c implies that there exists a basic Vietoris neighborhood $\langle V_1, ..., V_k \rangle$ of $E \cup \{x,y\}$ such that for every $F \in \langle V_1, ..., V_k \rangle \cap \mathcal{J}_n(X) \Rightarrow c(F) \in V$, therefore $c(F) \neq x$. Let $i_1, ..., i_j$ be indices for which $V_{i_k} \cap E \neq \emptyset$. Clearly, $\left\langle V_{i_1}, ..., V_{i_j} \right\rangle$ is a Vietoris neighborhood of E, satisfying $F \cup \{x,y\} \in \langle V_1, ..., V_k \rangle \cap \mathcal{J}_n(X)$ for every $F \in \left\langle V_{i_1}, ..., V_{i_j} \right\rangle \cap \mathcal{J}_{n-2}(X)$. Thus we obtained that for every F having the above property $c(F \cup \{x,y\}) \neq x$.

Assume $x \notin E$ and $c(E \cup \{x,y\}) = x$. Let us consider disjoint open sets U, V for which $E \cup \{y\} \subseteq V$, $x \in U$. Since c is continuous there is Vietoris neighborhood $\langle V_1, ..., V_k \rangle$ of $E \cup \{x,y\}$ such that whenever $F \in \langle V_1, ..., V_k \rangle \cap \mathcal{J}_n(X)$ then $c(F) \in U$. Let $i_1, ..., i_j$ be those indices for which V_{i_k} and E have common element. In this case $\left\langle V_{i_1} \cap V, ..., V_{i_j} \cap V \right\rangle$ is a Vietoris neighborhood of E that satisfies that $c(F \cup \{x,y\}) \in U$ for every $F \in \left\langle V_{i_1} \cap V, ..., V_{i_j} \cap V \right\rangle \cap \mathcal{J}_{n-2}(X)$. This x is the only element of the set $F \cup \{x,y\}$ which belongs to U as well, therefore $c(F \cup \{x,y\}) = x$ for every $F \in \left\langle V_{i_1} \cap V, ..., V_{i_j} \cap V \right\rangle \cap \mathcal{J}_{n-2}(X)$.
$\square$

Proposition 3. *Let (X,τ) be a connected and separated topological space, $\mathcal{J}_n(X)$ supplied with the Vietoris topology, and $n \geq 2$. If $c : \mathcal{J}_n(X) \to X$ is a continuous choice function then WARP holds, and the strict revealed relation P is open.*

Proof. First, we have to verify that the relation P is asymmetric. It is known (see Theorem 4.10 in Michael (1951), that the connectedness of τ implies the connectedness of Vietoris topology on $\mathcal{J}_n(X)$. It is sufficient to show that $c(E) = c(\{c(E), y\})$ for every $y \in E \in \mathcal{J}_n(X)$, $y \neq c(E)$. For, if $(x,y) \in P$ and $(y,x) \in P$ both hold then $x \neq y$ and there are sets $S_1, S_2 \in \mathcal{J}_n(X)$ such that $\{x,y\} \subseteq S_1 \cap S_2$ and $x = c(S_1)$, but $y = c(S_2)$. Therefore $x = c(\{c(S_1), y\}) = c(\{x,y\}) = c(\{x, c(S_2)\}) = y$, a contradiction.

We are going to verify that $c(E) = c(\{c(E), y\})$ for every $y \in E$ and $y \neq c(E)$ by induction. If E has only two elements then this statement is obviously true. Let us assume that the equation holds for any set containing at most $m-1$ elements, $3 \leq m \leq n$. Let $E = \{x_1, ..., x_m\}$ be a set with exactly m elements and $c(E) = x_1$. We show $c(\{x_1, x_i\}) = x_1$ if $i > 1$. Suppose the contrary: if there exists k $(1 < k \leq m)$ for which $c(\{x_1, x_k\}) = x_k$ then let us consider the following set.

$$\mathcal{H} \doteq \{F \in \mathcal{J}_{m-2}(X) : c(F \cup \{x_1, x_k\}) = x_1\}$$

Clearly, $E \setminus \{x_1, x_k\} \in \mathcal{H}$ and $\{x_1, x_k\} \notin \mathcal{H}$. By Proposition 2 the complement of $\mathcal{H}$ is an open subset of $\mathcal{J}_{m-2}(X)$. If $F \in \mathcal{H}$ and x_1 were an element of F then we would obtain that $c(\{x_1, x_k\}) = x_1$ by the the induction hypothesis for the $m-1$ element set $F \cup \{x_1, x_k\}$ and $c(F \cup \{x_1, x_k\}) = x_1$ contradicting $c(\{x_1, x_k\}) = x_k$. Therefore

$$\mathcal{H} = \{F \in \mathcal{J}_{m-2}(X) : x_1 \notin F \text{ and } c(F \cup \{x_1, x_k\}) = x_1\}.$$

Applying Proposition 2 again we obtain that $\mathcal{H}$ is also a nonempty open subset of $\mathcal{J}_{m-2}(X)$, which contradicts the connectedness of the Vietoris topology on $\mathcal{J}_{m-2}(X)$.

Second, we have to verify that the relation P is open. Let $(x, y) \in P$. WARP says that $c(\{x, y\}) = x$. Choose an open neighborhood $U \in \tau(x)$ with $y \notin U$. There exist neighborhoods $U' \in \tau(x)$ and $V' \in \tau(y)$ such that $c(F) \subseteq U$ for any set $F \in \langle U', V' \rangle$ by the continuity of c. For an arbitrary element $x' \in U' \cap U$ set $F \doteq \{x', y\}$, so we have $c(\{x', y\}) = x'$. This proves the openness of the upper level set P^y. The verification of the openness of the lower level sets is analogous. $\square$

A counterexample shows the necessity of connectedness. Let $X = (0,1) \cup \{2\}$ and τ be the relative topology from the Euclidean topology. Let the choice $c : \mathcal{J}_3(X) \to X$ be defined by

$$c(A) \doteq \begin{cases} \min A & \text{, if } 2 \in A \\ \max A & \text{, otherwise} \end{cases}$$

On the one hand it is clear that WARP fails in the generated decision structure. On the other hand, if $\{x, y, 2\} \in \mathcal{J}_3(X)$ then there exists a Vietoris neighborhood $\langle V_1, V_2, V_3 \rangle$ of $\{x, y, 2\}$, such that $2 \in A$ for every $A \in \mathcal{J}_3(X) \cap \langle V_1, V_2, V_3 \rangle$. This property implies the continuity of c.

Remark 4. *Next, we show that Proposition 3 remains valid if we replace the domain of choice with a set $\mathcal{D}$ such that $\mathcal{J}(X) \subseteq \mathcal{D}$.*

Proof. Denote by P_n, P' and P the strict revealed preference relation generated by the decision structure $(X, \mathcal{J}_n(X), c)$, $(X, \mathcal{J}(X), c)$ and $(X, \mathcal{D}, c)$, respectively. We already know that P_n is asymmetric and open. It is easy to check, that $P' = \cup_n P_n$, and $P_n \subseteq P_{n+1}$ for every n. This implies that the relation P' is also asymmetric and open. Verifying the asymmetry and the openness of the relation P we should verify again the equation $c(E) = c(\{c(E), y\})$ for every $y \in E \in \mathcal{D}$, as long as $c : \mathcal{D} \to X$ is continuous. Suppose the contrary: if $x = c(E), y \in E$ and $c(\{x, y\}) = y$ then $(y, x) \in P'$, that is $x \in P'_y$. The

set P'_y is an open neighborhood of x, therefore there exists a Vietoris neighborhood $\langle V_1, \dots, V_n \rangle$ of E with the property $F \in \langle V_1, \dots, V_n \rangle \cap \mathcal{B} \Rightarrow c(F) \in P'_y$. Thus, taking a finite set $F \in \mathcal{J}(X)$ such that $y \in F \in \langle V_1, \dots, V_n \rangle$, the inclusion $c(F) \in P'_y$ holds. On the one hand, the definition of the lower level set says $(y, z) \in P'$, where $z \doteq c(F)$. On the other hand, the finiteness of F and the definition of the relation P' implies that $(z, y) \in P'$, which contradicts the asymmetric property of P'. The rest of the proof of Proposition 3 is as above.
$\square$

The following proposition is the converse of the previous one in some sense and belongs to the problem drawn up in "general framework".

To explain why Vietoris topologies are natural to use, consider the classical consumer decision-making problem in which continuity of the demand function is vital. However, if the demand correspondence is in fact set-valued, a topology on the subsets has to be used. Our definition of upper hemicontinuity (through Vietoris topologies) is identical to the usual (see Hildenbrand (1974)).

If we introduce the budget correspondence $\beta(p) \doteq \left\{x \in \mathbb{R}^n_+ : p \cdot x \leq 1\right\}$ (which is Vietoris continuous at a strictly positive price vector) then the demand correspondence is the composition of a choice correspondence and the budget correspondence (i.e. $d = c \circ \beta$). Therefore, the upper hemicontinuity of the demand correspondence is a straightforward consequence of the continuity of choice, endowing the domain with Vietoris topology, and supplying the range with upper Vietoris topology.

At this point our paper is related to the celebrated article of Uzawa (1960). It is well-known that *if a compact subset $K \in \mathcal{K}(X)$ is given, and R is a transitive and complete relation with the openness of its asymmetric part then K has at least one greatest element with respect to R* that is, the chosen set $c^R(K)$ is nonempty. This fact means that all compact sets belong to the set $\mathcal{B}$ defined below, and the subsequent theorem is a generalization of results on the continuity of the demand correspondence. Note that Uzawa exploits an additional convexity assumption for deriving the continuity of demand function. (See Theorem 6 in Uzawa (1960)).

Proposition 5. *Let R be a transitive, complete relation and suppose that $A(R)_x$ and $A(R)^x$ are open level sets. Define the subset of the power set of alternatives*

$$\mathcal{B} \doteq \left\{B \subseteq X : B \cap F \neq \emptyset \Rightarrow c^R(B \cap F) \neq \emptyset \ \forall F \in \mathcal{F}\right\},$$

where $\mathcal{F}$ denotes all closed subsets of X. Then we obtain a decision structure $\left(X, \mathcal{B}, c^R\right)$ where the correspondence c^R is upper hemicontinuous.

Proof. It is easy to see that $\left(X, \mathcal{B}, c^R\right)$ is a choice structure satisfying WCA (R.3 and R.1). Let $E \in \mathcal{B}$ be a fixed set and $U \in \tau$ such that $c^R(E) \subseteq U$. We exhibit a Vietoris neighborhood of E, for which any set F from this neighborhood $c^R(F) \subseteq U$. When $E \subseteq U$ then U^+ is suitable, because $F \in U^+ \Rightarrow c^R(F) \subseteq F \subseteq U$. It can be assumed that $E \cap U^c \neq \emptyset$ in the rest of this proof. By the definition of $\mathcal{B}$ there is element y belonging to the set $c^R(E \cap U^c)$.

Clearly, $(x,y) \in R$ for any fixed $x \in c^R(E)$. If (y,x) were also an element of R then $y \in c^R(E) \subseteq U$ would be satisfied by the WCA, contradicting $y \in U^c$. Therefore $(x,y) \in A(R)$.

First we assume that there is no $z \in X$, for which $(x,z) \in A(R)$ and $(z,y) \in A(R)$. In this case let $N_1 \doteq A(R)^y \cap U$, $N_2 \doteq A(R)_x$. Obviously N_1 and N_2 are open sets, such that $x \in N_1$, $y \in N_2$ and hence $E \in N_1^- \cap N_2^-$. We verify that $E \subseteq N_1 \cup N_2$ is also satisfied. $E \cap A(R)^y \subseteq U$ holds by the choice of y. We know that the transitivity and the completeness of R imply negative transitivity of relation $A(R)$. From the negative transitivity of the $A(R)$ follows $N_2 \cup A(R)^y = X$. Hence we have

$$E \subseteq N_2 \cup (E \cap A(R)^y) \subseteq N_2 \cup (A(R)^y \cap U) = N_2 \cup N_1.$$

Thus $\langle N_1, N_2 \rangle$ is a Vietoris neighborhood of E. If $F \in \langle N_1, N_2 \rangle \cap \mathcal{B}$, and furthermore $t_1 \in F \cap N_1$ and $t_2 \in F \cap N_2$ then $(t_2, t_1) \notin R$ for in the opposite case both t_1 and t_2 would also be suitable for such a point z which satisfies $(x,z) \in A(R)$ and $(z,y) \in A(R)$. We obtain that $c^R(F) \cap N_2 = \emptyset$ and, consequently, $c^R(F) \subseteq N_1 \subseteq U$ holds for any $F \in \langle N_1, N_2 \rangle \cap \mathcal{B}$.

Second, we suppose that there exists a point z, such that $(x,z) \in A(R)$ and $(z,y) \in A(R)$. In this case let $N_1 \doteq U$, $N_2 \doteq U \cap A(R)^z$, $N_3 \doteq A(R)_z$. Clearly, these are open sets having elements in common with E because $x \in N_1$, $x \in N_2$, $y \in N_3$. We verify again that $E \subseteq N_1 \cup N_2 \cup N_3$ is satisfied. Let us observe that $A(R)^z \cap E \subseteq A(R)^y \cap E \subseteq U$ and $S(R)_z \cap E \subseteq A(R)^y \cap E \subseteq U$, because of the transitivity of R. Hence

$$\begin{aligned} E &\subseteq A(R)_z \cup (A(R)^z \cap E) \cup (S(R)_z \cap E) \\ &\subseteq A(R)_z \cup (A(R)^z \cap U) \cup U = N_3 \cup N_2 \cup N_1. \end{aligned}$$

Thus $\langle N_1, N_2, N_3 \rangle$ is really a Vietoris neighborhood of E. Take an arbitrary element F from the set $\langle N_1, N_2, N_3 \rangle \cap \mathcal{B}$. If $t_2 \in N_2$ and $t_3 \in N_3$ then $(t_2, t_3) \in A(R)$ that is, $(t_3, t_2) \notin R$. From this it follows that $c^R(F) \cap N_3 = \emptyset$ by the definition of c^R. Therefore $c^R(F) \subseteq N_1 \cup N_2 \subseteq U$. □

Let us consider a choice correspondence c satisfying WARP with domain $\mathcal{D}$, where $\mathcal{J}_3(X) \subseteq \mathcal{D} \subseteq \mathcal{K}(X)$. The condition that the domain includes all sets with at most three elements assures the transitivity and the completeness of the weak revealed relation R and $c(B) = c^R(B)$ for any $B \in \mathcal{D}$ (R.4). Clearly, $\mathcal{D} \subseteq \mathcal{B}$ also holds, where $\mathcal{B}$ is defined as in the previous theorem, because WARP implies the equation $P = A(R)$ (R.2), thus the asymmetric part of the weak revealed relation is open, supposing that the strict revealed relation is open. Hence, the following proposition is a corollary of Proposition 5.

Proposition 6. *Let $\mathcal{J}_3(X) \subseteq \mathcal{D} \subseteq \mathcal{K}(X)$ be the domain of the choice correspondence $c : \mathcal{D} \to \mathcal{P}(X)$ which satisfies WARP and the strict revealed relation P be open. Then c is upper hemicontinuous.*

Turning back to the "special framework" we are ready to present the proof of the main result.

Proof of Theorem 1. The special case of Proposition 6 says that if $c : \mathcal{J}_n(X) \to X$ a choice function, $n \geq 3$, WARP holds true, and the revealed preference P is open then c is continuous. The proof of the converse statement is given in Proposition 3. $\square$

Let us consider again the problem which was originally formulated by Arrow (1959).

Remark 7. *If the domain of choice $\mathcal{D}$ satisfies $\mathcal{J}(X) \subseteq \mathcal{D} \subseteq \mathcal{K}(X)$ and $c : \mathcal{D} \to X$ is a continuous choice function then the special case of Proposition 6 and Remark 4 show that Theorem 1 remains true.*

Finally, a counterexample shows that Theorem 1 fails in the case when the choice function is actually set-valued. If $\mathcal{J}_3(\mathbb{R})$ carries the Vietoris topology and $c : \mathcal{J}_3(\mathbb{R}) \to \mathcal{J}_3(\mathbb{R})$ is the choice function defined by $c(A) \doteq \{\max(A), \min(A)\}$ for every set A with at most three elements then c is continuous. Nevertheless WCA does not hold: indeed $1, 2 \in c(\{1,2\})$ and $2 \in c(\{0,1,2\})$ but $1 \notin c(\{0,1,2\})$.

We have seen that bringing the results of Michael's paper into view that had become the base of the set valued topology applied in numerous areas can lead to new results in the field of revealed preference theory.

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