Revealed Preference Tests of the Cournot Model

Andres Carvajal, Rahul Deb, James Fenske, and John K.-H. Quah

Background: Afriat's Theorem

Suppose we have a set $\mathcal{T} = \{1, 2, ..., T\}$ of observations drawn from a single consumer. Each observation consists of a price vector $p_t = (p_t^1, p_t^2, ..., p_t^l)$ and a consumption bundle $x_t = (x_t^1, x_t^2, ..., x_t^l)$ chosen by the consumer at p_t.

When are the observations $\{(p_t, x_t)\}_{t \in \mathcal{T}}$ consistent with a utility-maximizing consumer?

Formally, we wish to test the hypothesis H: there exists an increasing function $U : R_+^l \to R$ such that $x_t = \text{argmax}_{B_t} U(x)$, where

$$B_t = \{\bar{x} \in R_+^l : p_t \cdot \bar{x} \leq p_t \cdot x_t\}.$$

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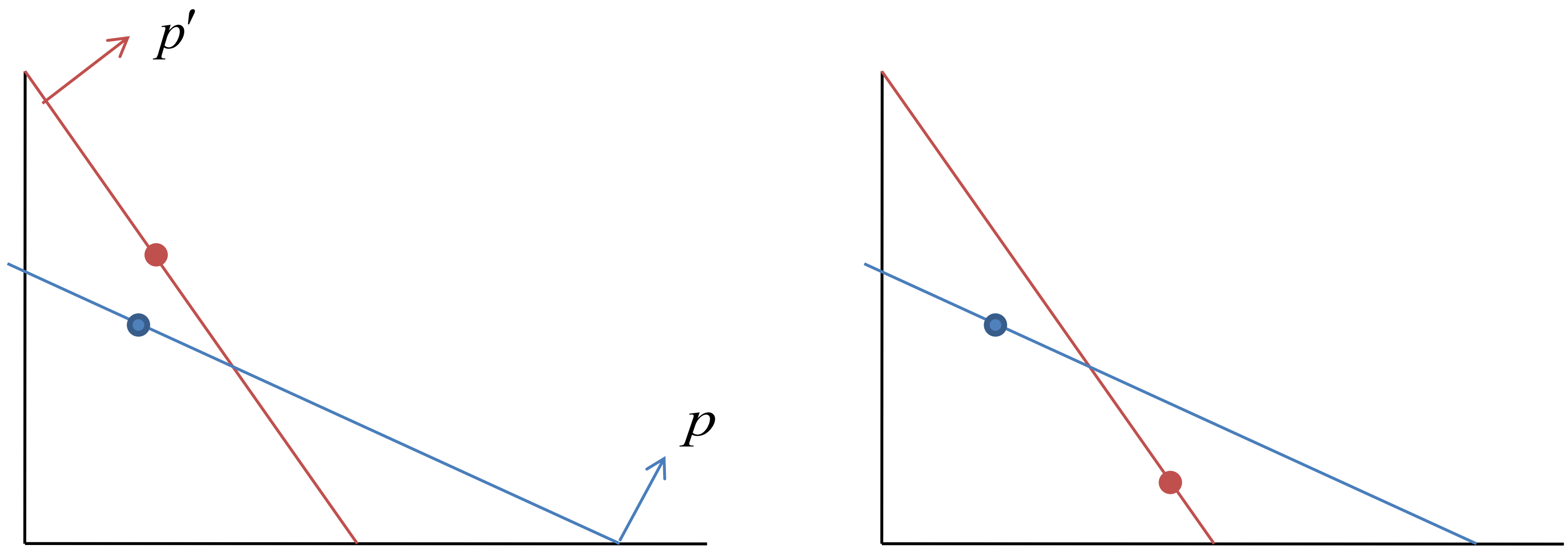
Afriat's Theorem: The set of observations $\{(p_t, x_t)\}_{t \in \mathcal{T}}$ is consistent with H if and only if it obeys the generalized axiom of revealed preference (GARP).

Background: Afriat's Theorem

Afriat's Theorem: The set of observations $\{(p_t, x_t)\}_{t \in \mathcal{T}}$ is consistent with H if and only if it obeys GARP.

Various generalizations of Afriat's result (by Varian and many other authors) and also applications to data.

Example of GARP violation:



Revealed preference in the Cournot model

Suppose we have a set $\mathcal{T} = \{1, 2, ..., T\}$ of observations drawn from an industry producing a homogeneous good. Each observation consists of the market price p_t and the output vector $(q_{i,t})_{i \in I}$, where I is the set of firms and $q_{i,t}$ is the output of firm i in observation t.

Suppose that firms' cost functions are unchanged across observations and the observations are generated by changes to the market demand function.

In this case, what restrictions on the data would we expect, if any?

Revealed preference in the Cournot model

Suppose that at observation t, the market inverse demand function is $\bar{P}_t$. Then the first order condition for profit maximization for firm i is

$$C_i'(q_{i,t}) = \bar{P}_t(Q_t) + q_{i,t}\bar{P}_t'(Q_t)$$

where $q_{i,t}$ is the output of firm i and $Q_t = \sum_{i \in I} q_{i,t}$ is the total output.

Re-arranging, we obtain

$$-\bar{P}_t'(Q_t) = \frac{\bar{P}_t(Q_t) - C_1'(q_{1,t})}{q_{1,t}} = \frac{\bar{P}_t(Q_t) - C_2'(q_{2,t})}{q_{2,t}} = \ldots = \frac{\bar{P}_t(Q_t) - C_I'(q_{I,t})}{q_{I,t}}$$

This implies that if $q_{i,t} > q_{j,t}$ then $C_i'(q_{i,t}) < C_j'(q_{j,t})$. In other words, *a firm with the larger share has the lower marginal cost.*

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This implies that if $q_{i,t} > q_{j,t}$ then $C_i'(q_{i,t}) < C_j'(q_{j,t})$. In other words, *a firm with the larger share has the lower marginal cost.*

Conclusion: if every firm has constant marginal costs (i.e., constant with respect to its output), then their *rank* cannot change across observations.

Revealed preference in the Cournot model

A similar observable restriction holds when firms have increasing marginal costs.

Suppose at observation t, firm i produces 20 and firm j produces 15. At another observation t', firm i produces 15 and firm j produces 16.

This is *not rationalizable* with a Cournot model with increasing marginal costs.

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Suppose at observation t, firm i produces 20 and firm j produces 15. At another observation t', firm i produces 15 and firm j produces 16.

This is *not rationalizable* with a Cournot model with increasing marginal costs.

Proof: Observation t tells us that $C_i'(20) < C_j'(15)$.

If firm i and j both have increasing marginal costs then

$$C_i'(15) \leq C_i'(20) < C_j'(15) \leq C_j(16).$$

But observation t' tells us that $C_i'(15) > C_j'(16)$. *QED*

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But observation t' tells us that $C_i'(15) > C_j'(16)$. *QED*

Note: the restriction does not even rely on price information!

Revealed preference in the Cournot model

Definition: A set of observations is $\{[p_t, (q_{i,t})_{i\in I}]\}_{t\in\mathcal{T}}$ is rationalizable with a Cournot model with constant (increasing) marginal costs if there are linear (convex) cost functions $\bar{C}_i : \mathbb{R}_+ \to \mathbb{R}$ for each firm $i \in I$ and downward sloping inverse demand functions $\bar{P}_t : \mathbb{R}_+ \to \mathbb{R}$ for each $t \in \mathcal{T}$, such that

(i) $\bar{P}_t(Q_t) = P_t$; and

(ii) $\text{argmax}_{\tilde{q}_i\geq 0}[\tilde{q}_i\bar{P}_t(\tilde{q}_i + \sum_{j\neq i} q_{j,t}) - C_i(\tilde{q}_i)] = q_{i,t}$.

Note: Condition (i) says that the inverse demand functions agree with the observed price and industry output at each observation.

Condition (ii) says that, at each observation t, firm i's observed output level $q_{i,t}$ maximizes its profit given the output of the other firms.

Revealed preference in the Cournot model

Recall that

$$-\bar{P}'_t(Q_t) = \frac{\bar{P}_t(Q_t) - C'_1(q_{1,t})}{q_{1,t}} = \frac{\bar{P}_t(Q_t) - C'_2(q_{2,t})}{q_{2,t}} = \ldots = \frac{\bar{P}_t(Q_t) - C'_I(q_{I,t})}{q_{I,t}}.$$

Clearly, if $\{[p_t, (q_{i,t})_{i \in I}]\}_{t \in \mathcal{T}}$ is compatible with a Cournot model with constant marginal costs then there must be $\{\lambda_i\}_{i \in I} \gg 0$ such that,

$$0 < \frac{p_t - \lambda_1}{q_{1,t}} = \frac{p_t - \lambda_2}{q_{2,t}} = \ldots = \frac{p_t - \lambda_I}{q_{I,t}} \text{ for all } t \in \mathcal{T}. \qquad (1)$$

Theorem 1: A set of observations is $\{[p_t, (q_{i,t})_{i \in I}]\}_{t \in \mathcal{T}}$ is rationalizable with a Cournot model with constant marginal costs if and only if there exists $\{\lambda_i\}_{i \in I} \gg 0$ such that (1) is satisfied.

Revealed preference in the Cournot model

$$-\bar{P}'_t(Q_t) = \frac{\bar{P}_t(Q_t) - C'_1(q_{1,t})}{q_{1,t}} = \frac{\bar{P}_t(Q_t) - C'_2(q_{2,t})}{q_{2,t}} = \ldots = \frac{\bar{P}_t(Q_t) - C'_I(q_{I,t})}{q_{I,t}}.$$

If $\{[p_t, (q_{i,t})_{i \in I}]\}_{t \in \mathcal{T}}$ is compatible with a Cournot model with increasing marginal costs then the following must be satisfied:

[A] there exists $\{\lambda_{i,t}\}_{(i,t) \in I \times \mathcal{T}}$ such that,

$$0 < \frac{p_t - \lambda_{1,t}}{q_{1,t}} = \frac{p_t - \lambda_{2,t}}{q_{2,t}} = \ldots = \frac{p_t - \lambda_{I,t}}{q_{I,t}} \text{ for all } t \in \mathcal{T}$$

(we refer to this as the common ratio property) and

[B] for each firm i, the coefficients $\{\lambda_{i,t}\}_{t \in \mathcal{T}}$ are co-monotonic with its output, i.e.,

$$\lambda_{i,t} \geq \lambda_{i,t'} \text{ if } q_{i,t} > q_{i,t'}.$$

Revealed preference in the Cournot model

Theorem 2: A set of observations is $\left\{[p_t, (q_{i,t})]_{i \in I}\right\}_{t \in \mathcal{T}}$ is rationalizable with a Cournot model with increasing marginal costs if and only if conditions [A] and [B] are satisfied.

Revealed preference in the Cournot model

Theorem 2: A set of observations is $\left\{[p_t, (q_{i,t})]_{i \in I}\right\}_{t \in \mathcal{T}}$ is rationalizable with a Cournot model with increasing marginal costs if and only if conditions [A] and [B] are satisfied.

Proof of sufficiency: Condition [A] says there exists $\{\lambda_{i,t}\}_{(i,t) \in I \times \mathcal{T}}$ such that,

$$0 < \frac{p_t - \lambda_{1,t}}{q_{1,t}} = \frac{p_t - \lambda_{2,t}}{q_{2,t}} = \ldots = \frac{p_t - \lambda_{I,t}}{q_{I,t}} \text{ for all } t \in \mathcal{T}. \tag{3}$$

Construct a cost function for firm i such that $\bar{C}_i'(q_{i,t}) = \lambda_{i,t}$. Because of condition [B], $\bar{C}_i$ can be chosen to have increasing marginal cost.

For each t, let the demand function be $\bar{P}_t(Q) = a_t - b_t Q$, where $b_t = \frac{(p_t - \lambda_{i,t})}{q_{i,t}}$ and choose a_t to solve $a_t - b_t Q_t = p_t$, so $\bar{P}_t$ is compatible with the observation at t.

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Assume that for every $j \neq i$, firm j's output is $q_{j,t}$. The firm i's chooses $\tilde{q}_i$ to maximize $\tilde{q}_i \bar{P}_t(\tilde{q}_i + \sum_{j\neq i} q_{j,t}) - C_i(\tilde{q}_i)$. The first order condition for this problem is $-\tilde{q}_i b_t + \left[a_t - b_t(\tilde{q}_i + \sum_{j\neq i} q_{j,t})\right] - C_i'(\tilde{q}_i) = 0$. This is satisfied at $\tilde{q}_i = q_{i,t}$ because $-q_{i,t} b_t + p_t - \lambda_{i,t} = 0$.

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Construct a cost function for firm i such that $\bar{C}_i'(q_{i,t}) = \lambda_{i,t}$. Because of condition [B], $\bar{C}_i$ can be chosen to have increasing marginal cost.

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The multi-product Cournot model

Suppose there are N products (forming the set $\mathcal{N}$) in this industry. The clearing price of good k depends on the total output *vector* $Q \in R^N_+$, so we write it as $\bar{P}^k(Q)$.

The inverse demand system $P = (P^k)_{k \in \mathcal{N}}$ obeys the law of demand if

$$\text{the derivative matrix } \left[\frac{\partial P^k}{\partial Q^\ell}(Q)\right]_{(k,\ell)\in\mathcal{N}\times\mathcal{N}}$$

is negative definite for all Q. This is equivalent to

$$(Q - Q') \cdot (P(Q) - P(Q')) < 0 \text{ for distinct } Q \text{ and } Q'.$$

Firm i's cost depends on the output vector it chooses to produce; formally, we denote the cost of producing $q_i \in R^N_+$ by $C_i(q_i)$.

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Observation t consists of the price vector $p_t = (p^k_t)_{k \in \mathcal{N}}$ and output vectors $q^t_i \in R^N_+$, for each firm i.

What conditions are necessary for $\left\{[p_t, (q_{i,t})]_{i \in I}\right\}_{t \in \mathcal{T}}$ to be Cournot rationalizable?

The multi-product Cournot model

Theorem 3: A set of observations is $\left\{[p_t, (q_{i,t})]_{i \in I}\right\}_{t \in \mathcal{T}}$ is rationalizable with a Cournot model with convex cost functions for every firm and inverse demand functions $\bar{P}_t^k$ obeying the law of demand if and only if there exists a solution to a system of polynomial inequalities with parameters derived from the data.

In fact, like the single good case, the inverse demand function for each good can be chosen to be linear in the output vector Q whenever a rationalization exists.

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$$\frac{\partial \bar{P}_t^k}{\partial Q^j} \leq 0 \text{ for all } j \in \mathcal{N} \text{ and } k \in \mathcal{N}$$

if and only if there exists $\mu_t^k \in R_+^N$, for $k = 1, 2, .., N$, and $C_{i,t}$ for all $(i, t) \in I \times \mathcal{T}$ such that

(i) $C_{i,t'} \geq C_{i,t} + \sum_{k=1}^N (p_t^k - \mu_t^k \cdot q_{i,t})(q_{i,t'}^k - q_{i,t}^k)$ and

(ii) $p_t^k - \mu_t^k \cdot q_{i,t} > 0$ for all $i \in I$ and $k \in \mathcal{N}$.

Revealed preference in the Cournot model

What if marginal cost is not increasing?

Proposition: Any set of observations $\left\{[p_t,(q_{i,t})]_{i\in I}\right\}_{t\in\mathcal{T}}$ is rationalizable with a Cournot model (with not necessarily increasing marginal costs).

Revealed preference in the Cournot model

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Proposition: Any set of observations $\left\{[p_t, (q_{i,t})]_{i\in I}\right\}_{t\in\mathcal{T}}$ is rationalizable with a Cournot model (with not necessarily increasing marginal costs).

“Proof": First we find $\{\lambda_{i,t}\}_{(i,t)\in I\times\mathcal{T}}$ that satisfies the common ratio property:

$$0 < \frac{p_t - \lambda_{1,t}}{q_{1,t}} = \frac{p_t - \lambda_{2,t}}{q_{2,t}} = \ldots = \frac{p_t - \lambda_{I,t}}{q_{I,t}} \quad \text{for all } t \in \mathcal{T} \tag{7}$$

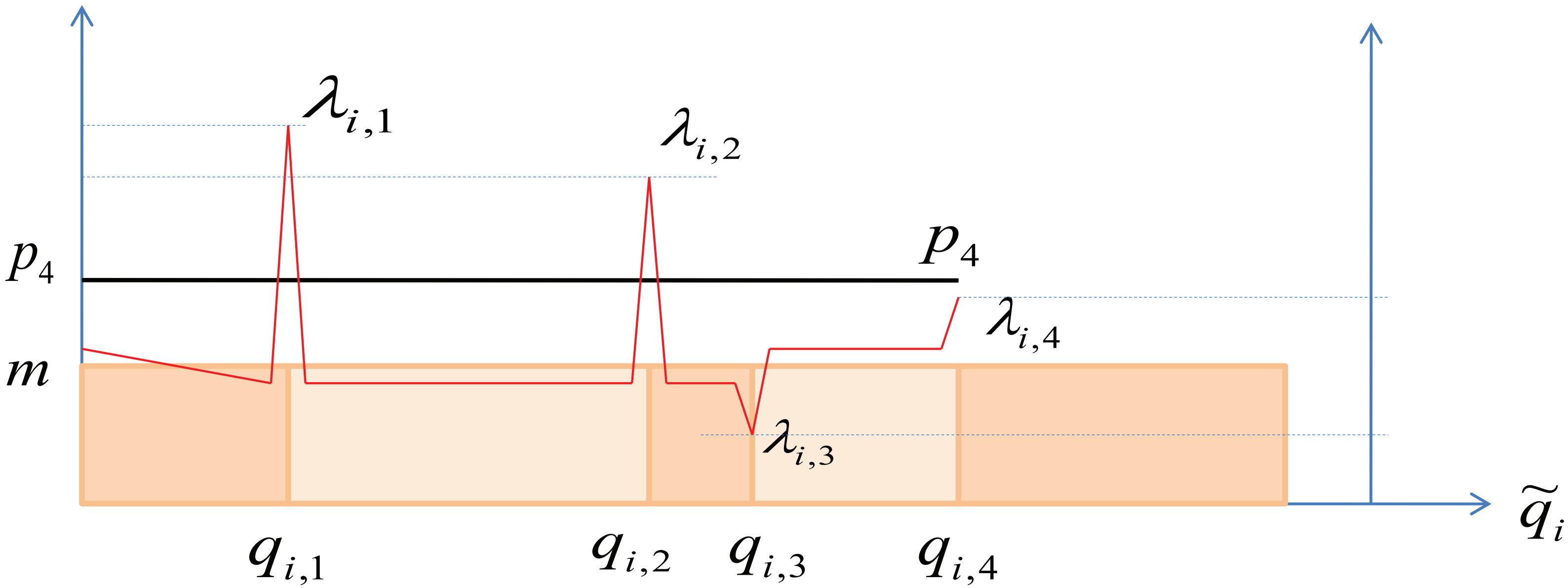
This is *always* possible.

After that we construct cost functions C_i (for each firm i), with $C_i'(q_{i,t}) = \lambda_{i,t}$, and inverse demand functions $\bar{P}_t$ (for each $t \in \mathcal{T}$) such that $q_{i,t}$ maximizes firm i's profit at observation t.

This is possible because, loosely speaking, cost could be chosen to be arbitrarily low and the function $\bar{P}_t$ could be chosen to have a steep fall at Q_t.

Revealed preference in the Cournot model

First choose m such that $0 < m < p_t$ for all t. After that fit in a marginal cost curve such that $C_i'(q_{i,t}) = \lambda_{i,t}$ and $C_i(q_{i,t}) = mq_{i,t}$ for all $q_{i,t}$. Because we do not restrict the shape of the marginal cost curve, this is always possible.



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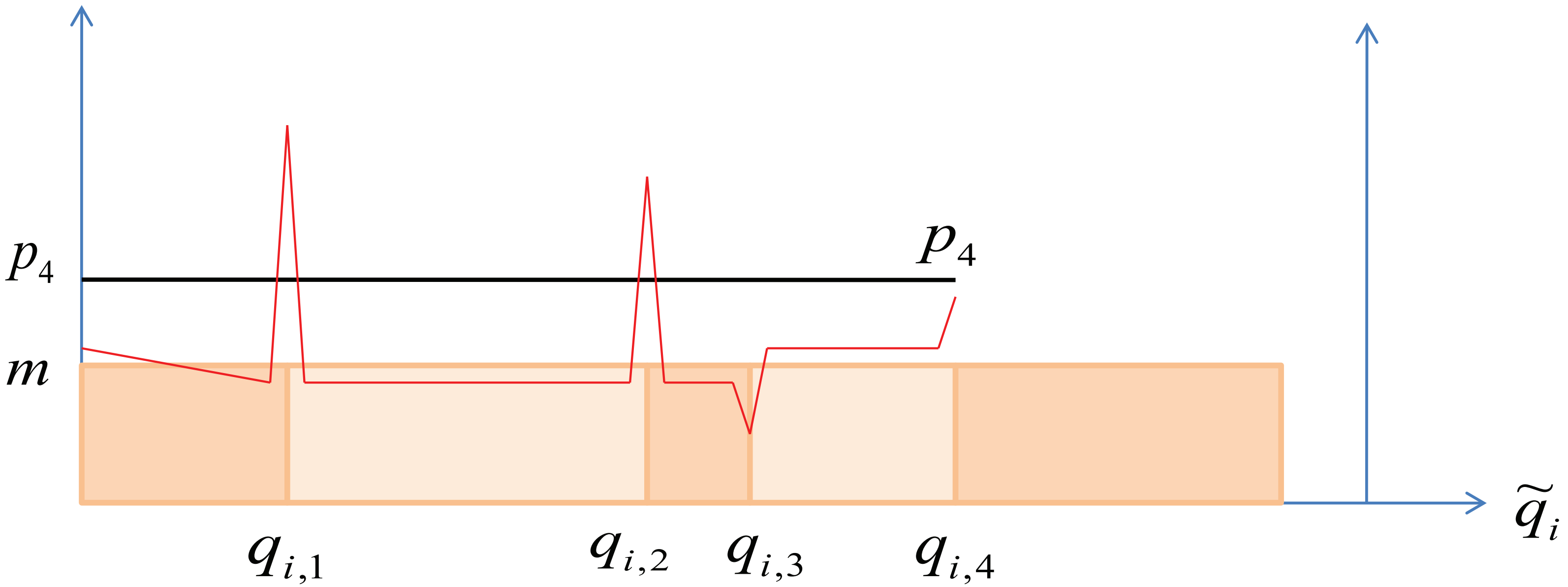

For all $\tilde{q} < q_{i,4}$, we have $\int_{\tilde{q}_i}^{q_{i,4}} C_i'(q)\,dq \approx m(q_{i,4} - \tilde{q}_i) < p_4(q_{i,4} - \tilde{q}_i)$.

Revealed preference in the Cournot model

The reason for the indeterminacy result is that observed market shares no longer convey non-infinitesimal information about *discrete* marginal costs.

When marginal costs are increasing, if firm i produces 20 and firm j produces 15 at some observation, then $C_i'(20) < C_j'(15)$ and thus

$$C_i'(q_i) < C_j'(q_j) \text{ for all } q_i < 20 \text{ and } q_j > 15.$$

A point observation conveys information on marginal costs over entire intervals.

If marginal cost curves are completely arbitrary, then $C_i'(20) < C_j'(15)$ says just that.

This permissiveness is too extreme.

Revealed preference in the Cournot model

To restore the connection between infinitesimal and discrete marginal costs, we do the following.

Let $q_{i,\ell(t)}$ be the observed output level of firm i immediately below $q_{i,t}$. We require the marginal cost of increasing output from $q_{i,\ell(t)}$ to $q_{i,t}$ to be at least $\Delta_{i,t} = \frac{1}{2}\left[\delta_{i,t} + \delta_{i,\ell(t)}\right]$. We refer to this as the convincing criterion.

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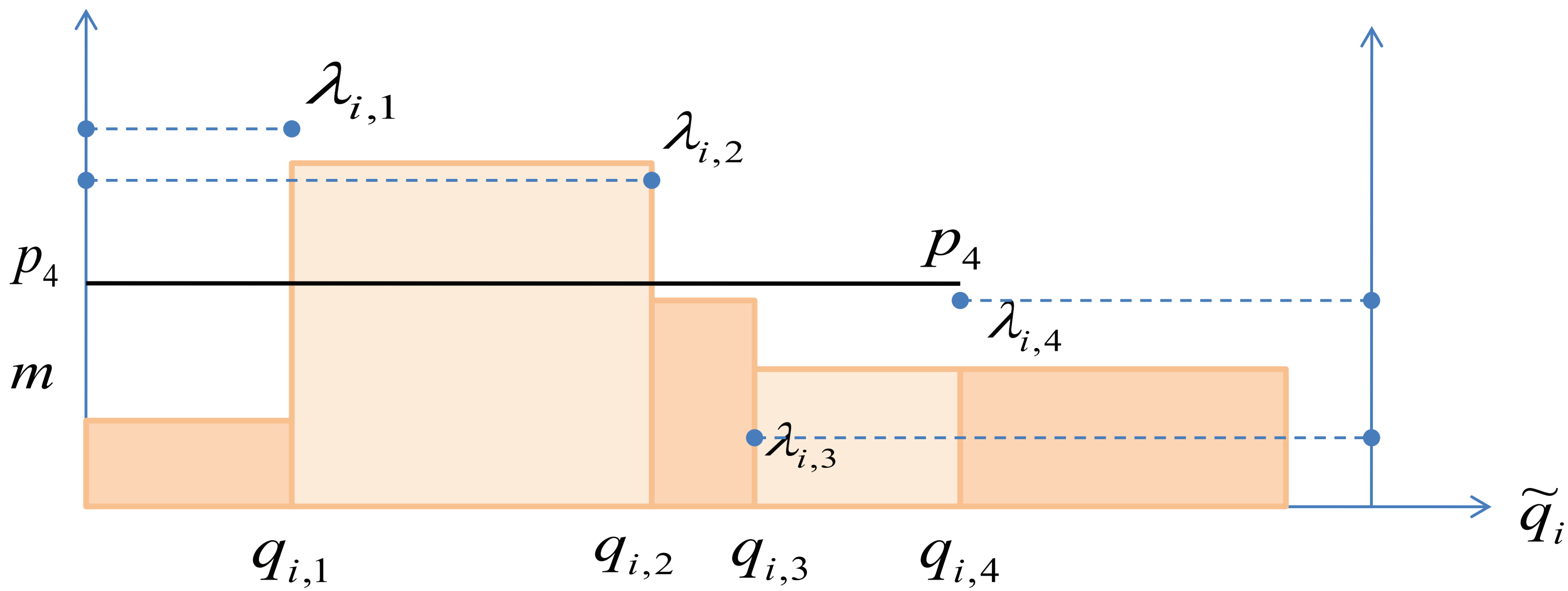
Now global optimality is not trivially true any more ...



Revealed preference in the Cournot model

Theorem 4: A set of observations is $\left\{[p_t, (q_{i,t})]_{i\in I}\right\}_{t\in\mathcal{T}}$ is convincingly rationalizable with a Cournot model if and only if the following conditions are satisfied:

[A] there exists $\{\lambda_{i,t}\}_{(i,t)\in I\times\mathcal{T}}$ such that,

$$0 < \frac{p_t - \lambda_{1,t}}{q_{1,t}} = \frac{p_t - \lambda_{2,t}}{q_{2,t}} = \ldots = \frac{p_t - \lambda_{I,t}}{q_{I,t}} \text{ for all } t \in \mathcal{T} \qquad (8)$$

(this is the common ratio property).

[B] Let $\Delta_{i,t} = \frac{1}{2}\left[\delta_{i,t} + \delta_{i,\ell(t)}\right]$ and denote the set of observations t' such that $q_{i,t'} < q_{i,t}$ by $L_i(t)$. Then for each firm i and observation t,

$$\sum_{s\in(L_i(t)\cup\{t\})\setminus L_i(t')} \Delta_{i,s}\left(q_{i,s} - q_{i,l(s)}\right) < p_t(q_{i,t} - q_{i,t'}) \text{ for } t' \in L_i(t). \qquad (9)$$

We refer to (9) as the discrete marginal property.

Collusion and Conjectural Variations

Let $\left\{[p_t, (q_{i,t})]_{i\in I}\right\}_{t\in\mathcal{T}}$ be a set of observations.

Choose a number m such that $0 < m < p_t$ for all t. Then it is clear that there are inverse demand functions $\bar{P}_t$, with $\bar{P}_t(Q_t) = p_t$ such that

$$Q_t = \text{argmax}_{\tilde{q}>0}\tilde{q}\bar{P}_t(\tilde{q}) - m\tilde{q}.$$

In other words, *any* set of observations is consistent with collusion. So while Cournot interaction is refutable, collusion is not.

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Definition: Suppose the inverse demand function is $\bar{P}_t$. Then $(q_{i,t})_{i\in I}$ constitutes a θ-CV equilibrium (with $\theta=\{\theta_i\}_{i\in\mathcal{I}}\gg 0$) if

$$\text{argmax}_{\tilde{q}_i\geq 0}\left[\tilde{q}_i\bar{P}_t\left(\theta_i(\tilde{q}_i-q_{i,t})+Q_t\right)-\bar{C}_i(\tilde{q}_i)\right]=q_{i,t}.$$

If firms in the industry are price-taking, then $\theta=(0,0,...,0)$.
A Cournot equilibrium is a θ-CV equilibrium with $\theta=(1,1,...,1)$.
Collusion corresponds to the case where $\theta>(1,1,...,1)$.

Collusion and Conjectural Variations

Definition: A set of observations $\{[p_t, (q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is θ-CV rationalizable (with $\theta = \{\theta_i\}_{i\in\mathcal{I}} \gg 0$) if we can find a downward sloping demand function, $\bar{P}_t$, for each observation t, and cost functions, $\bar{C}_i$, for each firm i, such that
(i) $\bar{P}_t(\sum_{i\in\mathcal{I}} q_{i,t}) = p_t$ and
(ii) $(q_{i,t})_{i\in I}$ obeys $\text{argmax}_{\tilde{q}_i\geq 0}\left[\tilde{q}_i\bar{P}_t\left(\theta_i(\tilde{q}_i - q_{i,t}) + Q_t\right) - \bar{C}_i(\tilde{q}_i)\right] = q_{i,t}$.

The first order condition for firm i, at the equilibrium is

$$\bar{P}_t(Q_t) + \theta_i q_{i,t}\bar{P}_t'(Q_t) - \bar{C}_i'(q_{i,t}) = 0.$$

So we obtain

$$-\bar{P}_t'(Q_t) = \frac{p_t - \bar{C}_1'(q_{1,t})}{\theta_1\, q_{1,t}} = \frac{p_t - \bar{C}_2'(q_{2,t})}{\theta_2\, q_{2,t}} = \ldots = \frac{p_t - \bar{C}_I'(q_{I,t})}{\theta_I\, q_{I,t}}.$$

Collusion and Conjectural Variations

Theorem 5: A set of observations is $\left\{[p_t, (q_{i,t})]_{i \in I}\right\}_{t \in \mathcal{T}}$ is $\{\theta_i\}_{i \in \mathcal{I}}$-CV rationalizable with increasing marginal costs if and only if the following hold:

[A] there exists $\{\lambda_{i,t}\}_{(i,t) \in I \times \mathcal{T}}$ such that

$$0 < \frac{p_t - \lambda_{1,t}}{\theta_1 q_{1,t}} = \frac{p_t - \lambda_{2,t}}{\theta_2 q_{2,t}} = \ldots = \frac{p_t - \lambda_{I,t}}{\theta_I q_{I,t}} \text{ for all } t \in \mathcal{T}$$

[B] for each firm i, the coefficients $\{\lambda_{i,t}\}_{t \in \mathcal{T}}$ obey: $\lambda_{i,t} \geq \lambda_{i,t'}$ if $q_{i,t} > q_{i,t'}$.

Corollary: A set of observations is Cournot rationalizable if and only if it is $(\bar{\theta}, \bar{\theta}, .., \bar{\theta})$-CV rationalizable for any positive scalar $\bar{\theta}$.

Collusion and Conjectural Variations

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Corollary: A set of observations is Cournot rationalizable if and only if it is $(\bar{\theta}, \bar{\theta}, .., \bar{\theta})$-CV rationalizable for any positive scalar $\bar{\theta}$.

In this sense, a rejection of the Cournot model in this context is very strong because it is equivalent to a rejection of every symmetric-CV model.

Collusion and Conjectural Variations

Information on demand allows for sharper restrictions on θ.

Example: Consider a duopoly with firms i and j where
(i) at observation t, $P_t = 10$, $Q_{i,t} = 5/3$ and $Q_{j,t} = 5/3$; and
(ii) at observation t', $P_{t'} = 4$, $Q_{i,t'} = 2$ and $Q_{j,t'} = 5/3$.
In addition, suppose $d\bar{P}_t/dq \geq -3$.

These observations are compatible with $\theta = (3,3)$ but not $\theta = (1,1)$.

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Suppose the data set is Cournot rationalizable with a rationalizing demand $\bar{P}_t$ satisfying $d\bar{P}_t/dq \geq -3$. From the first order condition for firm i

$$\frac{10 - m_{i,t}}{5/3} = -\frac{d\bar{P}_t}{dq} \leq 3,$$

where $m_{i,t}$ is a subgradient of firm i's cost function at output $Q_{i,t} = 5/3$. Therefore, $m_{i,t} \geq 5$.

This means that the marginal cost at $q_{i,t'} = 2$ must be at least 5 since firm i's cost function is convex. However, the price at t' is just 4, so there is a contradiction.

The Cournot hypothesis in the oil market

Testing Cournot rationalizability with convex costs

Rejection Rates: OPEC

		Number of Countries			
		2	3	6	12
Window	3 Months	0.28	0.54	0.89	1.00
	6 Months	0.65	0.89	1.00	1.00
	12 Months	0.90	0.99	1.00	1.00

Rejection Rates: Non-OPEC

		Number of Countries			
		2	3	6	9
Window	3 Months	0.46	0.77	0.99	1.00
	6 Months	0.85	0.98	1.00	1.00
	12 Months	0.97	1.00	1.00	1.00

The rejection rate reported is the proportion of cases that were rejected.

For example, there are 434 three month periods in the data. There are 66 possible combinations of two out of twelve OPEC members. The entry for two countries and three months reports that out of the 434X66=28,644 tests of two OPEC members over three months, 8138, or 28% could not be rationalized.

The Cournot hypothesis in the oil market

Testing Convincing Cournot rationalizability

Rejection Rates: OPEC

		Number of Countries			
		2	3	6	12
	3 Months	0.21	0.41	0.76	0.98
Window	6 Months	0.40	0.66	0.92	1.00
	12 Months	0.60	0.84	0.98	1.00

Rejection Rates: Non-OPEC

		Number of Countries			
		2	3	6	9
	3 Months	0.07	0.16	0.44	0.70
Window	6 Months	0.15	0.32	0.74	0.98
	12 Months	0.26	0.51	0.91	1.00

Revealed preference with changing cost functions

It is possible to allow for the possibility that firms' cost *functions* may vary across observations.

In addition to prices and firm-level outputs, suppose the observer observes parameter α_i that has an impact on firm i's cost function, which we denote as $\bar{C}_i(\cdot;\alpha_i)$.

Assume that α_i is drawn from a partially ordered set (for example, some subset of the Euclidean space endowed with the product order). Firm i has a differentiable cost function and higher values of α_i lead to higher marginal costs; i.e., if $\bar{\alpha}_i > \hat{\alpha}_i$, then

$$\bar{C}_i'(q_i;\bar{\alpha}_i) \geq C_i'(q_i;\hat{\alpha}_i) \text{ for all } q_i > 0.$$

For example, α_i could be the observable price of some input in the production process.

A set of observations takes the form $\{[p_t,(q_{i,t})_{i\in\mathcal{I}},(a_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$.

Revealed preference with changing cost functions

Definition: $\{[p_t, (q_{i,t})_{i\in\mathcal{I}}, (a_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is Cournot rationalizable with C^2 and convex cost functions that agree with $\{a_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ if there exist C^2 and convex cost functions $\bar{C}_i(\cdot; a_{i,t})$ (for each firm i at observation t), and downward sloping demand functions $\bar{P}_t$ for each observation t such that

(i) $\bar{P}_t(Q_t) = p_t$;

(ii) $q_{i,t} \in \text{argmax}_{\bar{q}_i \geq 0} \left\{ \bar{q}_i \bar{P}_t(\bar{q}_i + \sum_{j\neq i} q_{j,t}) - \bar{C}_i(\bar{q}_i; a_{i,t}) \right\}$; and

(iii) $\bar{C}_i'(\cdot; a_{i,t}) \geq \bar{C}_i'(\cdot; a_{i,\tilde{t}})$ if $a_{i,t} > a_{i,\tilde{t}}$ and $\bar{C}_i'(\cdot; a_{i,t}) = \bar{C}_i'(\cdot; a_{i,\tilde{t}})$ if $a_{i,t} = a_{i,\tilde{t}}$.

Revealed preference with changing cost functions

Theorem: The following statements on $\{[p_t, (q_{i,t})_{i\in\mathcal{I}}, (a_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ are equivalent.

[A] The set of observations is Cournot rationalizable with C^2 and convex cost functions that agree with $\{a_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$.

[B] There exists a set of positive scalars $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ satisfying the common ratio property, with

$$\delta_{i,t'} \geq (=)\, \delta_{i,t} \text{ whenever } q_{i,t'} \geq (=)\, q_{i,t} \text{ and } a_{i,t'} \geq (=)\, a_{i,t}.$$