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REVEALED PREFERENCE TESTS OF THE COURNOT MODEL

By Andres Carvajal, Rahul Deb, James Fenske, and John K.-H. Quah

Abstract: We consider an observer who makes a finite number of observations of an industry producing a homogeneous good, where each observation consists of the market price and firm-specific production quantities. We develop a revealed preference test (in the form of a linear program) for the hypothesis that the firms are playing a Cournot game, assuming that they have convex cost functions that do not change and the observations are generated by the demand function varying across observations. Extending this basic result, we develop tests for the case where (in addition to changes to demand) firms' cost functions may vary across observations. We also develop tests of Cournot interaction in cases where there are multiple products and where cost functions may be non-convex. Applying these results to the crude oil market, we show that Cournot behavior is strongly rejected.

Keywords: nonparametric test, observable restrictions, linear programming, multi-product Cournot oligopoly, collusion, crude oil market

JEL Codes: C14, C61, C72, D21, D43

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1. Introduction

Consider a finite set of observations where each observation consists of a price vector (representing the prices of m goods) and a demand bundle. In an influential paper, Afriat (1967) posed the following question: what restrictions on this data set are necessary and sufficient for it to be consistent with observations drawn from a utility-maximizing consumer? Through his work and that of others, it is now well-known that the condition required is the generalized axiom of revealed preference (or GARP for short).[1] A large literature on consumer behavior - both theoretical and empirical - has been built on Afriat's theorem.

A natural extension of this question is to derive testable restrictions on outcomes in a general equilibrium setting. This question was first posed and answered by Brown and Matzkin (1996), who considered a set of observations drawn from an exchange economy, where each observation consists of the aggregate endowment, the income distribution, and an equilibrium price vector. They found testable conditions under which these observations are consistent with Walrasian equilibria in an exchange economy where endowments are changing across observations and utility functions are held fixed. As only aggregate consumption (equivalently, endowment) data is observed, the issue is whether this could be split into individual consumption bundles such that each agent in the economy is utility-maximizing with respect to the given market prices. To show that observable restrictions exist, Brown and Matzkin also provide an example of a data set that is not consistent with Walrasian outcomes.

In this paper, we ask a question similar in spirit to the one posed by Brown and Matzkin but in a multi-agent game theoretic setting. We consider a finite set of observations, $\mathcal{T}$, of an industry producing a single good; each observation t in $\mathcal{T}$ consists of the price the good P_t and the output of each firm, so $Q_{i,t}$ is the output of firm i (in the set $\mathcal{I}$). The set of observations can thus be written as $\{P_t, (Q_{i,t})_{i\in\mathcal{I}}\}_{t\in\mathcal{T}}$. We ask two related questions. (1) Are there any observable restrictions implied by the following hypothesis: that each observation in the data set is a Cournot equilibrium, assuming that each firm has a convex cost function that does not vary across observations and that the data is generated by changes to a downward-sloping demand function? (2) If the answer to the first question is 'yes', precisely what conditions must a data set obey for it to be consistent with this hypothesis? In other words, what restrictions on the data set are necessary and sufficient for the existence of a cost function for each firm and demand functions at each observation

[1] The term was introduced by Varian (1982). For a discussion of results closely related to Afriat's theorem, by Samuelson, Houthakker and other authors, see Mas-Colell et al. (1995).

that will generate the observed data as Cournot equilibria?

In Section 2 of this paper, we show that there *are* observable restrictions and that the hypothesis is satisfied if and only if there is a solution to a particular linear program constructed from the data set. Whether or not the latter condition holds can be resolved in finitely many steps, so the problem of whether the hypothesis is satisfied is a solvable problem. We show, in addition, that these results can be extended to allow the firms' cost functions to vary across observations.

In Section 4 of the paper, we generalize this result to a market consisting of several goods, with each firm producing one or more of the goods; in this context we work out precise observable restrictions that have to be satisfied for a data set to be consistent with a multi-product Cournot equilibrium, with firms having convex cost functions and demand obeying the law of demand and varying across observations.

There have been few attempts to derive revealed preference tests for game theory models and models with externalities in general. A possible reason is that the presence of externalities implies that the set of potential preferences for each agent could be very large. As a result, one would expect the testable restrictions imposed by these models to be extremely weak and hence uninteresting. In light of this, the fact that there are any observable restrictions at all in our setup may seem surprising, so it is worth giving some intuition. Consider, once again, the single-product case and assume that each firm's cost curves are not just convex but linear, so that firms can be ranked by their marginal costs. It is well-known (and trivial to show) that firm's market shares are ranked inversely with their marginal costs. This is true whatever the demand function. It follows that any data set in which firms are observed to *change ranks* within the data set is not consistent with Cournot outcomes and firms having constant marginal costs. When costs are convex rather than linear, firm rankings can change within a data set, so the observable restrictions are more subtle, but they still exist.

This naturally raises the question of whether there are any restrictions on the data set if firms are allowed to have non-convex cost functions (maintaining the assumption that cost functions do not vary across observations in the data set). We show that the answer to this question is 'no'; any data set is Cournot rationalizable if firms' marginal cost functions can be drawn from the set of all continuous (but not necessarily increasing) positive-valued functions.

This result is not as negative as it seems, because a natural bound on the 'wriggliness' of the marginal cost function, which we call the *convincing criterion*, is sufficient to restore the refutability of the model. Let $Q_{i,t'}$ and $Q_{i,t''}$ be two neighboring observed output choices of firm i (in other words, no other observed outputs of firm i lie between $Q_{i,t'}$ and $Q_{i,t''}$). A rationalizing cost function $\bar{C}_i$ for firm i is said to satisfy the convincing criterion if the average marginal cost between these observations is at least as great as $0.5[\bar{C}_i'(Q_{i,t'}) + \bar{C}_i'(Q_{i,t''})]$. Note that the convincing criterion is not a restriction on the shape of the marginal cost curve that is independent of the observed outputs; instead, it forbids the modeler from choosing as the rationalizing cost function for a firm i a function where the average marginal cost between two observed outputs is lower than the infinitesimal marginal costs at either observed output. Put another way, the convincing criterion requires that the marginal cost information gleaned from the output observations must convey some information about marginal costs *between* observations.[2]

In Section 6 we show that Cournot rationalizability with convincing cost functions holds if and only if there is a solution to a particular linear program. The possibility of non-convex costs means that the firm's profit maximization problem need not be quasiconcave. This makes our result quite unusual since most revealed preference models typically rely on convex analysis and so rely, in some form, on concavity or convexity assumptions.[3] Similarly, econometric analyses that recover model parameters (like the degree of competitiveness, see below) through first order conditions also rely on the quasi-concavity of the optimization problem; otherwise, there is no guarantee that observations satisfying the first order conditions are globally optimal.

It is worth emphasizing that our principal objective is specifically to devise a test for the Cournot model in which rationalizing cost functions can be chosen from a very large class and no assumptions are made on the evolution of demand across observations. In particular, we are not principally concerned with detecting collusion or the degree of collusion (variously measured). In fact, the absence of any information (hence restrictions) on how demand varies in our setup means that *any* data set is consistent with perfect collusion amongst firms (see Section 3).

This observation is consistent with the results of Bresnahan (1982) and Lau (1982), who show that identifying the degree of competitiveness of a firm (in a conjectural variations model) requires an observer to know how demand changes with some observable parameters and also that the changes be generated by at least a two-parameter family; hence empirical IO studies that address this question will typically estimate the demand

[2]Notice that this issue does not arise when marginal costs are increasing so it does not have to be explicitly addressed. In that case, the infinitesimal marginal cost at some output level is a lower bound on marginal costs at *all* higher output levels.

[3]There are exceptions, including Matzkin (1991) and, more recently, Forges and Minelli (2008) who consider the possibility of non-convex budget sets in the consumer problem.

function alongside estimating the degree of competitiveness. Loosely speaking, the tests we develop avoid having to do this by exploiting the fact that while the degree of competitiveness of a particular firm cannot be determined without greater information on demand, the Cournot equilibrium implies that this degree of competitiveness is *equal amongst firms* and equality can be tested without information on demand. It is possible to extend our methods to measure the degree of competitiveness (rather than simply testing the Cournot hypothesis) but in keeping with the Bresnahan-Lau results, more information is required; for example, bounds on the price elasticity of demand at each observation will lead to bounds on the degree of competitiveness for each firm. These issues are formally addressed in Section 3.

Given that we assume we have no information on costs or demand (and therefore impose very few restrictions on either), the tests of the Cournot model we have constructed seem very permissive and it is not clear that they have the power to reject real data. So as a simple application, we apply our tests to the oil-producing countries both within and outside of OPEC. Our task is made easy by the fact that the tests take the form of checking whether a linear program admits a solution; in this regard, it is different from Brown and Matzkin's test of the Walrasian hypothesis, the implementation of which is complicated by the fact that it involves the computationally far more demanding task of checking for a solution to a system of polynomial inequalities. We tested for Cournot rationalizability with convex cost functions and also with convincing cost functions. The former hypothesis is clearly rejected by the data. With the latter the outcome is more mixed, but it is clear that this test is also discriminating.

Related literature. Brown and Matzkin's result in the context of exchange economies has been extended in a number of ways to take into account of (for example) financial markets (Kubler, 2003), random preferences (Carvajal, 2004), and externalities (Carvajal (2009) and Deb (2009)).

It is also natural to investigate the testable implications of games. Sprumont (2000) considers this question in the context of normal form games and asks when observed actions can be rationalized as Nash equilbria. Ray and Zhou (2001) address the same question for extensive form games. These papers differ from our work in two critical ways. Firstly, in their work, payoff functions remain fixed and the variability in the data arises from players choosing actions from different subsets of their strategies across observations. Hence, their results are not applicable to our context, where it is the payoff functions that are changing across observations (because of changes to demand).

A second difference is that they develop their results in a context where game outcomes at *all* subsets of strategies are known. In formal terms, they consider a situation where a map from subsets of strategies to outcomes is observed. They identify the necessary and sufficient restrictions that such a map must satisfy for it to be considered an equilibrium map, i.e., a map from the strategy subset to a Nash equilibrium for that strategy subset. Clearly, any restrictions found in such a context must remain necessary when this map is only partially known (in the sense that one knows the outcomes at some but not all strategy subsets), but they may no longer be sufficient (see, for example, Section 4.1 in Sprumont (2000)). The problem we consider is analogous to the case where only part of this map is known, since we observe industry outcomes for some but not all possible demand functions. Nonetheless, it is possible to obtain observable restrictions that are not just necessary, but also sufficient, for Cournot rationalizability.[4]

Partly motivated by earlier versions of this paper, Routledge (2009) has provided a revealed preference analysis of the Bertrand game. It is clear that many extensions and variations on this theme are possible and worth studying, and also empirical work that can be done based on this approach.

2. Cournot Rationalizability

An industry consists of I firms producing a homogeneous good; we denote the set of firms by $\mathcal{I} = \{1, 2, \ldots, I\}$. Consider an experiment in which T observations are made of this industry. We index the observations by $t \in \mathcal{T} = \{1, 2, \ldots, T\}$. For each t, the industry price P_t and the output of each firm $(Q_{i,t})_{i\in\mathcal{I}}$ are observed; we require $Q_{i,t} > 0$ for all (i, t). The aggregate output of the industry at observation t is denoted by $Q_t = \sum_{i\in\mathcal{I}} Q_{i,t}$.

We say that the set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is *Cournot rationalizable* if each observation can be explained as a Cournot equilibrium arising from a different market demand function, keeping the cost function of each firm fixed across observations, and with the demand and cost functions obeying certain regularity properties. By a *cost function* of firm i we mean a strictly increasing function $\bar{C}_i : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\bar{C}_i(0) = 0$. The market inverse demand function $\bar{P}_t : \mathbb{R}_+ \to \mathbb{R}$ (for each t) is said to be *downward sloping* if it is differentiable at any $q > 0$, with $\bar{P}'_t(q) < 0$. For example, the linear inverse demand function given by $\bar{P}_t(q) = a_t - b_t q$, for $a_t > 0$ and $b_t > 0$ is downward sloping in our sense.

[4] A similar distinction exists in demand theory, between rationalizability results where only the demand at some price vectors are observed (like Afriat's theorem) and results which assume that the entire demand function is observed (see the discussion in Afriat (1967)).

Formally, $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is Cournot rationalizable if there exist cost functions $\bar{C}_i$ for each firm i and downward sloping demand functions $\bar{P}_t$ for each observation t such that
(i) $\bar{P}_t(Q_t) = P_t$; and
(ii) $Q_{i,t} \in \text{argmax}_{q_i \geq 0} \left\{ q_i \bar{P}_t(q_i + \sum_{j\neq i} Q_{j,t}) - \bar{C}_i(q_i) \right\}$.
Condition (i) says that the inverse demand function must agree with the observed data at each t. Condition (ii) says that, at each observation t, firm i's observed output level $Q_{i,t}$ maximizes its profit given the output of the other firms. Note that in any Cournot rationalizable data set, the observed prices P_t must be strictly positive. This is because we assume that observed output is nonzero and firms' costs are strictly increasing in output; if $P_t \leq 0$, a firm would be strictly better off producing nothing.

A standard assumption made in theoretical and econometric work is that cost functions are convex. This assumption is often made because it helps to make the optimization problem tractable and in many settings it is not an implausible assumption. Our main goal in this section is to determine the precise conditions under which a set of observations is Cournot rationalizable with convex cost functions. It is not immediately obvious that such a condition imposes *any* restrictions on the data, so we should first demonstrate that it does.

Suppose $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is rationalized by demand functions $\{\bar{P}_t\}_{t\in\mathcal{T}}$ and cost functions $\{\bar{C}_i\}_{i\in\mathcal{I}}$. At observation t, firm i chooses q_i to maximize its profit given the output of the other firms (see (ii) above); at its optimal choice $Q_{i,t}$, the first order condition must be satisfied. Hence there is $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$ (the set of subgradients of $\bar{C}_i$ at $Q_{i,t}$) such that

$$Q_{i,t}\bar{P}_t'(Q_t) + \bar{P}_t(Q_t) - \delta_{i,t} = Q_{i,t}\bar{P}_t'(Q_t) + P_t - \delta_{i,t} = 0.$$

It follows that $\delta_{i,t}$ must obey the following condition, which we shall refer to as the *common ratio property*: for every $t \in \mathcal{T}$,

$$\frac{P_t - \delta_{1,t}}{Q_{1,t}} = \frac{P_t - \delta_{2,t}}{Q_{2,t}} = \ldots = \frac{P_t - \delta_{I,t}}{Q_{I,t}} > 0. \tag{1}$$

This holds because the first order condition guarantees that $(P_t - \delta_{i,t})/Q_{i,t} = -\bar{P}_t'(Q_t)$ and the latter is positive and independent of i. With the common ratio property we could recover information about a firm's marginal cost without directly observing it; when combined with the convexity of the cost function, it allows us to conclude that certain observations are not rationalizable, as we show in the following examples.

Example 1. Suppose that at observation t, firm i produces 20 and firm j produces 15. At another observation t', firm i produces 15 and firm j produces 16. We claim that these observations are not Cournot rationalizable with convex cost functions. Suppose, to the contrary, that it is. In that case, observation t tells us that there is $\delta_{i,t} \in \bar{C}'_i(20)$ and $\delta_{j,t} \in \bar{C}'_j(15)$ such that $\delta_{i,t} < \delta_{j,t}$. In other words, the firm with the larger output has lower marginal cost, which is an immediate consequence of the common ratio property. At observation t', firm i produces 15, which is less than its output at t; since $\bar{C}_i$ is convex, $\bar{C}'_i(15) \leq \delta_{i,t}$ (by this we mean that $\delta_{i,t}$ is weakly greater than every element in $\bar{C}'_i(15)$). Similarly, the convexity of $\bar{C}_j$ guarantees that $\bar{C}'_j(16) \geq \delta_{j,t}$ since firm j's output at t' is higher than its output at t. Putting these together, we obtain

$$\bar{C}'_i(15) \leq \delta_{i,t} < \delta_{j,t} \leq \bar{C}'_j(16),$$

but this violates the common ratio property since it means that at observation t', firm j has larger output *and* higher marginal cost compared to i.

Notice that Example 1 does not even rely on price information, so the mere observation of firm-level outputs can, in principle, contradict the Cournot hypothesis. In this example, the firms change ranks - the larger firm becomes smaller in another observation and also the outputs of the two firms are not moving co-monotonically. The next example is one in which the firms do not switch ranks and output movements are co-monotonic, but it is still not Cournot rationalizable.

Example 2. Consider the following observations of two firms i and j:
(i) at observation t, $P_t = 10$, $Q_{i,t} = 50$ and $Q_{j,t} = 100$;
(ii) at observation t', $P_{t'} = 4$, $Q_{i,t'} = 60$ and $Q_{j,t'} = 110$.
We claim that these observations are not Cournot rationalizable with convex cost functions. Indeed, if they are, then there is $\delta_{i,t} \in \bar{C}'_i(Q_{i,t})$ and $\delta_{j,t} \in \bar{C}'_j(Q_{j,t})$ such that

$$\delta_{i,t} = P_t - [P_t - \delta_{j,t}] \frac{Q_{i,t}}{Q_{j,t}} \geq P_t \left[1 - \frac{Q_{i,t}}{Q_{j,t}}\right]. \tag{2}$$

The equation on the left follows from the common ratio property and the inequality from the assumption that marginal cost is positive. Substituting in the numbers given, we obtain $\delta_{i,t} \geq 5$, where $\delta_{i,t} \in \bar{C}'_i(50)$. Since firm i has increasing marginal costs, the marginal cost of increasing its output from 50 to 60 is at least $5 \times 10 = 50$. However, the marginal revenue for firm i of increasing its output from 50 to 60 at observation t' is no greater than $4 \times 10 = 40$ (since $P_{t'} = 4$ and demand is downward-sloping). Therefore, at observation t', firm i is better off producing 50 than 60 – it is not maximizing its profit.

The next theorem is the main result of this section and shows that a set of observations is Cournot rationalizable with convex cost functions if and only if there is a solution to a certain linear program constructed from the data.

THEOREM 1. *The following statements on* $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ *are equivalent.*
[A] *The set of observations is Cournot rationalizable with convex cost functions.*
[B] *There exists a set of positive numbers* $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ *satisfying the common ratio property (1) and such that, for each* i, $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ *is* increasing with $Q_{i,t}$ *in the sense that* $\delta_{i,t'} \geq \delta_{i,t}$ *whenever* $Q_{i,t'} > Q_{i,t}$.

It is worth pointing out that Theorem 1 is useful even in situations where the output of one or more firms is missing from the data set. This is because if all of the firms in an industry are playing a Cournot game, then any subset of firms whose outputs *are* observed must also be playing a Cournot game (against each other and with the residual demand function as their 'market' demand function), and the latter hypothesis can be tested using the theorem.[5]

Our proof of Theorem 1 uses two lemmas; the first one provides an explicit construction of the demand curve needed to rationalize the data at any observation t, while the second lemma provides a way of constructing a cost curve for each firm obeying stipulated conditions on marginal cost.

LEMMA 1. *Suppose that, at some observation* t, *there are positive scalars* $\{\delta_{i,t}\}_{i\in\mathcal{I}}$ *such that (1) is satisfied and that there are convex cost functions* $\bar{C}_i$ *with* $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$. *Then there exists a downward-sloping demand function* $\bar{P}_t$ *such that* $\bar{P}_t(Q_t) = P_t$ *and, with each firm* i *having the cost function* $\bar{C}_i$, $\{Q_{i,t}\}_{i\in\mathcal{I}}$ *constitutes a Cournot equilibrium.*

Proof: We define $\bar{P}_t$ by $\bar{P}_t(Q) = a_t - b_t Q$, where $b_t = [P_t - \delta_{i,t}]/Q_{i,t}$ – notice that this is well-defined because of (1) – and choosing a_t such that $\bar{P}_t(Q_t) = P_t$. Firm i's decision is to choose $q_i \geq 0$ to maximize $\Pi_{i,t}(q_i) = q_i \bar{P}_t(q_i + \sum_{j\neq i} Q_{j,t}) - \bar{C}_i(q_i)$. This function is concave, so an output level is optimal if and only if it obeys the first order condition. Since $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$ and since $\bar{P}_t'(Q_t) = -b_t$, a supergradient[6] of $\Pi_{i,t}$ at $Q_{i,t}$ is

$$Q_{i,t}\bar{P}_t'(Q_t) + \bar{P}_t(Q_t) - \delta_{i,t} = -Q_{i,t}\frac{[P_t - \delta_{i,t}]}{Q_{i,t}} + P_t - \delta_{i,t} = 0.$$

So we have shown that $Q_{i,t}$ is profit-maximizing for firm i at observation t. QED

[5] In this regard, it is quite different from the inequality conditions of Afriat's Theorem which, unless preferences are separable, become vacuous when there is missing data (see Varian, 1988).

[6] By a *supergradient* of a concave function F at a point, we mean the subgradient of the convex function $-F$ at the same point.

LEMMA 2. *Suppose that for some firm i, there are positive scalars $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ that are increasing with $Q_{i,t}$ (in the sense defined in Theorem 1). Then there exists a convex cost function $\bar{C}_i$ such that $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$.*

Proof: Define $\hat{Q} = \{q_i \in \mathbb{R}_+ : q_i = Q_{i,t} \text{ for some observation } t\}$; $\hat{Q}$ consists of those output levels actually chosen by firm i at some observation. Since $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ are increasing with $Q_{i,t}$ it is possible to construct a strictly positive and increasing function $\bar{m}_i : \mathbb{R}_+ \to \mathbb{R}$ with the following properties: (a) for any output $\hat{q} \in \hat{Q}$, set $\bar{m}_i(\hat{q}) = \max\{\delta_{i,t} : Q_{i,t} = \hat{q}\}$; (b) for any $\hat{q} \in \hat{Q}$, $\lim_{q\to\hat{q}^-} \bar{m}_i(q) = \min\{\delta_{i,t} : Q_{i,t} = \hat{q}\}$; and (c) $\bar{m}_i$ is continuous at all $q \notin \hat{Q}$. The function $\bar{m}_i$ is piecewise continuous with a discontinuity at $\hat{q} \in \hat{Q}$ if and only if the set $\{\delta_{i,t} : Q_{i,t} = \hat{q}\}$ is non-singleton. Define $\bar{C}_i : \mathbb{R} \to \mathbb{R}$ by

$$\bar{C}_i(q) = \int_0^q \bar{m}_i(s)\, ds. \tag{3}$$

This function is strictly increasing because $\bar{m}_i$ is strictly positive and it is convex because $\bar{m}_i$ is increasing. Lastly, (a) and (b) guarantee that $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$. QED

Proof of Theorem 1: To see that [A] implies [B], suppose that the data is rationalized with demand functions $\{\bar{P}_t\}_{t\in\mathcal{T}}$ and cost functions $\{\bar{C}_i\}_{i\in\mathcal{I}}$. We have already shown that the first order condition guarantees the existence of $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$ obeying the common ratio property (1). Since $\bar{C}_i$ is convex, $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ is increasing with $Q_{i,t}$.

The fact that [B] implies [A] is an immediate consequence of Lemmas 1 and 2. QED

Sometimes it is convenient to consider rationalizations where each firm's cost functions are differentiable (so kinks on the cost curves are not allowed). This can be characterized by strengthening the condition imposed on $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ in Theorem 1; we say that $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ is *finely increasing with* $Q_{i,t}$ if it is increasing and $\delta_{i,t'} = \delta_{i,t}$ whenever $Q_{i,t} = Q_{i,t'}$. We may also have reason to believe that some firm i in the industry has constant marginal costs and would like to confirm that the data supports that hypothesis. This can be checked by requiring $\delta_{i,t}$ to be independent of t. We state this formally in the next result, which is a straightforward variation on Theorem 1.

COROLLARY 1. *The following statements on $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ are equivalent.*
[A] *The set of observations is Cournot rationalizable with convex cost functions for all firms and with firms in $\mathcal{J} \subseteq \mathcal{I}$ having C^2 cost functions and firms in $\mathcal{J}' \subseteq \mathcal{J}$ having linear cost functions.*[7]

[7]It is clear from the proof that the cost functions could in fact be chosen to be differentiable to any

[B] *There exists a set of positive numbers* $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ *satisfying the common ratio property and the following:* (a) $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ *is increasing with* $Q_{i,t}$ *(for every firm* i*);* (b) *for a firm* $i \in \mathcal{J}$, $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ *is finely increasing with* $Q_{i,t}$*; and* (c) *for a firm* $i \in \mathcal{J}'$, $\delta_{i,t'} = \delta_{i,t}$ *for all* $t \in \mathcal{T}$.

Proof: To see that [A] implies [B], suppose that the data is rationalized with demand functions $\{\bar{P}_t\}_{t\in\mathcal{T}}$ and cost functions $\{\bar{C}_i\}_{i\in\mathcal{I}}$. We have already shown in Theorem 1 that the first order condition guarantees the existence of $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$ obeying the common ratio property and condition (a) (in statement [B] above). Condition (b) holds since for a firm in $\mathcal{J}$, $\bar{C}_i'(Q_{i,t})$ is unique, so clearly $\delta_{i,t'} = \delta_{i,t}$ whenever $Q_{i,t} = Q_{i,t'}$. Lastly, a firm in $\mathcal{J}'$ has constant marginal cost, so $\delta_{i,t}$ does not vary with t (condition (c)).

To see that [B] implies [A], first note that Lemma 2 can be strengthened to say that (I) if the positive scalars $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ are finely increasing with $Q_{i,t}$, $\bar{C}_i$ can be chosen to be a C^2 function and (II) if the positive scalars $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ are independent of $Q_{i,t}$, then $\bar{C}_i$ can be chosen to be linear. This is clear from the proof of Lemma 2 where the marginal cost function $\bar{m}_i$ can be chosen to be a smooth function if $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ are finely increasing with $Q_{i,t}$ and is a constant function if $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ is independent of t. Consequently, $\bar{C}_i$ (as defined by equation (3) is, respectively, C^2 and linear. It is clear that these supplementary observations, when combined with Lemmas 1 and 2 guarantee that [B] implies [A]. QED

It is sometimes convenient in applications to allow for the possibility that firms' cost *functions* may vary across observations. There are quite a few ways in which these effects could potentially be taken into account. We illustrate how this can be done with one method of allowing for cost changes that we think is intuitive and instructive.

Assume that, in addition to prices and firm-level outputs, the observer also observes some parameter α_i that has an impact on firm i's cost function, which we denote as $\bar{C}_i(\cdot;\alpha_i)$. We assume that α_i is drawn from a partially ordered set (for example, some subset of the Euclidean space endowed with the product order). The firm i has a differentiable cost function and higher values of α_i are assumed to lead to higher marginal costs; formally, if $\bar{\alpha}_i > \hat{\alpha}_i$, then $\bar{C}_i'(q_i;\bar{\alpha}_i) \geq C_i'(q_i;\hat{\alpha}_i)$ for all $q_i > 0$. For example, α_i could be the observable price of some input in the production process. It is well-known that marginal cost increases with input price if we make the reasonable assumption that the demand for this input (as a function of the output level) is normal. Note that when α_i is not scalar but a vector (for example, the prices of different inputs), then it is not

degree, but there's no particular need to go beyond C^2, which is sufficient to ensure the differentiability of the marginal cost function.

always the case that observed parameters are comparable. When that happens we allow the marginal cost function at each parameter observation to differ without being ordered.

In this context, a set of observations takes the form $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (a_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$, where $a_{i,t}$ is the observed value of α_i at observation t. As before, we assume that $P_t > 0$ and $Q_{i,t} > 0$ for all (i,t). We say that this data set is *Cournot rationalizable with C^2 and convex cost functions that agree with $\{a_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$* if there exist C^2 and convex cost functions $\bar{C}_i(\cdot; a_{i,t})$ (for each firm i at observation t), and downward sloping demand functions $\bar{P}_t$ for each observation t such that

(i) $\bar{P}_t(Q_t) = P_t$;

(ii) $Q_{i,t} \in \text{argmax}_{q_i\geq 0}\left\{q_i\bar{P}_t(q_i + \sum_{j\neq i} Q_{j,t}) - \bar{C}_i(q_i; a_{i,t})\right\}$; and

(iii) $\bar{C}_i'(\cdot; a_{i,t}) \geq \bar{C}_i'(\cdot; a_{i,\tilde{t}})$ if $a_{i,t} > a_{i,\tilde{t}}$ and $\bar{C}_i'(\cdot; a_{i,t}) = \bar{C}_i'(\cdot; a_{i,\tilde{t}})$ if $a_{i,t} = a_{i,\tilde{t}}$.

The next result shows the equivalence between rationalizability in this sense and the solution to a linear program.

Corollary 2. *The following statements on $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (a_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ are equivalent.*

[A] *The set of observations is Cournot rationalizable with C^2 and convex cost functions that agree with $\{a_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$.*

[B] *There exists a set of positive scalars $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ satisfying the common ratio property, with*

$$\delta_{i,t'} \geq (=)\, \delta_{i,t} \;\; \textit{whenever } Q_{i,t'} \geq (=)\, Q_{i,t} \textit{ and } a_{i,t'} \geq (=)\, a_{i,t}. \tag{4}$$

Proof: To show that [A] implies [B], let $\delta_{i,t} = \bar{C}_i'(Q_{i,t}; a_{i,t})$. Then the common ratio property follows from the first order condition. If $Q_{i,t'} = Q_{i,t}$ and $a_{i,t'} = a_{i,t}$, we have $\bar{C}_i'(Q_{i,t}; a_{i,t}) = \bar{C}_i'(Q_{i,t'}; a_{i,t'})$, so $\delta_{i,t} = \delta_{i,t'}$. If $Q_{i,t'} \geq Q_{i,t}$ and $a_{i,t'} \geq a_{i,t}$, we have

$$\bar{C}_i'(Q_{i,t'}; a_{i,t'}) \geq \bar{C}_i'(Q_{i,t}; a_{i,t'}) \geq \bar{C}_i'(Q_{i,t}; a_{i,t}),$$

where the first inequality follows from the convexity of $\bar{C}_i(\cdot; a_{i,t'})$ and the second from the requirement that marginal cost increases with the observed parameter. In other words, $\delta_{i,t'} \geq \delta_{i,t}$.

To show that [B] implies [A], choose positive scalars $d_{i,t,\tilde{t}}$ for every $(i,t,\tilde{t}) \in \mathcal{I}\times\mathcal{T}\times\mathcal{T}$ with the following properties: (a) $d_{i,t',t'} = \delta_{i,t'}$, (b) $d_{i,t'',\hat{t}} = d_{i,t',\tilde{t}}$ whenever $Q_{i,t''} = Q_{i,t'}$ and $a_{i,\hat{t}} = a_{i,\tilde{t}}$, (c) $d_{i,t'',\hat{t}} \geq d_{i,t',\tilde{t}}$ whenever $Q_{i,t''} > Q_{i,t'}$ and $a_{i,\hat{t}} = a_{i,\tilde{t}}$, and (d) $d_{i,t'',\hat{t}} \geq d_{i,t',\tilde{t}}$ whenever $Q_{i,t''} = Q_{i,t'}$ and $a_{i,\hat{t}} > a_{i,\tilde{t}}$. This is possible because of (4). Due to (b) and (c), there is a C^2 and and convex cost function $\bar{C}_i(\cdot; a_{i,\tilde{t}})$ with

$$\bar{C}_i'(Q_{i,t}; a_{i,\tilde{t}}) = d_{i,t,\tilde{t}}. \tag{5}$$

Furthermore, because of (d), we could choose $\bar{C}_i$ in such a way that $\bar{C}_i'(\cdot; a_{i,\hat{t}}) \geq \bar{C}_i'(\cdot; a_{i,\tilde{t}})$ if $a_{i,\hat{t}} > a_{i,\tilde{t}}$. (These claims follow from Lemma 2 and straightforward modifications of its proof.) Notice that equation (5) and property (a) tells us that $\bar{C}_i'(Q_{i,\tilde{t}}; a_{i,\tilde{t}}) = \delta_{i,\tilde{t}}$, so the common ratio property on $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ tells us that

$$\frac{P_{\tilde{t}} - \bar{C}_1'(Q_{1,\tilde{t}}; a_{1,\tilde{t}})}{Q_{1,\tilde{t}}} = \frac{P_{\tilde{t}} - \bar{C}_2'(Q_{2,\tilde{t}}; a_{2,\tilde{t}})}{Q_{2,\tilde{t}}} = \ldots = \frac{P_{\tilde{t}} - \bar{C}_I'(Q_{I,\tilde{t}}; a_{I,\tilde{t}})}{Q_{I,\tilde{t}}} > 0. \tag{6}$$

By Lemma 1, there exists a downward sloping demand function $\bar{P}_{\tilde{t}}$ such that $\bar{P}_{\tilde{t}}(Q_{\tilde{t}}) = P_{\tilde{t}}$ and, with each firm i having the cost function $\bar{C}_i(\cdot; a_{i,\tilde{t}})$, $\{Q_{i,\tilde{t}}\}_{i\in\mathcal{I}}$ constitutes a Cournot equilibrium. QED

3. Testing for Collusion

A major concern in the empirical IO literature is the detection of collusive behavior (e.g. Porter, 2005). This question is related to, but distinct from, the principal focus of our paper, which is to develop a revealed preference test for Cournot behavior. In this section, we shall explain this distinction and also consider what added information is needed in our framework to test for collusion if that is what we wish to do.

Recall our basic assumption that the data set is generated by the interaction of firms in an industry, with costs unchanged across observations and the demand fluctuating. In the last section, we asked what conditions are needed for a data set $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ to be Cournot rationalizable; similarly, we could ask what conditions are needed for it be consistent with collusion, in the sense of all firms acting in concert to maximize joint profit. This question admits a short answer: *any* data set is consistent with collusion. The simple proof below provides rationalizing cost functions for each firm that are linear and identical across firms, and rationalizing demand functions at each observation t that are also linear.

Proposition 1. *For any set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ with $P_t > 0$ for all t, there is $\epsilon > 0$ and downward-sloping inverse demand functions $\bar{P}_t : \mathbb{R}_+ \to \mathbb{R}$ for each t, such that, for every t,*

$$(Q_{i,t})_{i\in\mathcal{I}} \in \operatorname{argmax}_{(q_i)_{i\in\mathcal{I}}\geq 0} \left[\left(\sum_{i\in\mathcal{I}} q_i\right) \bar{P}_t\left(\sum_{i\in\mathcal{I}} q_i\right) - \epsilon \left(\sum_{i\in\mathcal{I}} q_i\right) \right].$$

Proof: Suppose that every firm has cost function $\bar{C}(q) = \epsilon q$. Then every output allocation is cost efficient and if firms are colluding they will act like a monopoly with the same cost function $\bar{C}$. Choose ϵ sufficiently small so that $P_t > \epsilon$ for all t. It is straightforward to check that there is a linear and downward-sloping inverse demand function $\bar{P}_t$ such that $\bar{P}_t(Q_t) = P_t$ and such that the marginal revenue at Q_t is ϵ. QED

This proposition says that we could not exclude the possibility of collusive behavior, at least not if we assume no information on firms' costs and no information about the evolution of the demand curve beyond the point observations (P_t, Q_t) made at each t.

This message is reinforced if we embed the Cournot model within a model of conjectural variations, which is commonly used in empirical estimates of market power (see Bresnahan (1989)). Consider an industry with I firms, where $\bar{P}$ is the inverse demand function and where firm i has the cost function $\bar{C}_i$. To each firm we associate a real number $\theta_i \geq 0$; the output vector $(Q_i^*)_{i\in\mathcal{I}}$ constitutes a $\theta = \{\theta_i\}_{i\in\mathcal{I}}$ *conjectural variations equilibrium* (or θ-CV equilibrium, for short) if

$$Q_i^* \in \text{argmax}_{q_i \geq 0} \left\{ q_i \bar{P}\left(\theta_i (q_i - Q_i^*) + \sum_{j\in\mathcal{I}} Q_j^* \right) - \bar{C}_i(q_i) \right\}.$$

It is clear from firm i's optimization problem that firm i believes that as it deviates from Q_i^*, total output will change by the deviation multiplied by the factor θ_i. If $\theta_i = 1$ for all i, then we have the Cournot model; if $\theta_i = 0$ then the firms are acting as though its output has no impact on total output, so it is a price-taker. More generally, high values of θ_i across firms are interpreted as firms acting less competitively. A significant literature in empirical IO seeks to measure the level of competitiveness amongst firms by measuring θ_i. These studies typically assume that θ_i is the same across firms though, in principle, a firm's belief about the impact of its behavior may well differ from that of another firm.

A set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is said to be *θ-CV rationalizable* if there exist cost functions $\bar{C}_i$ (for each firm $i \in \mathcal{I}$) and inverse demand functions $\bar{P}_t$ (at each observation t) such that $\bar{P}_t(Q_t) = P_t$ and $(Q_{i,t})_{i\in\mathcal{I}}$ constitutes a θ-CV equilibrium. The next result, which gives a linear program to test for θ-CV rationalizability (for a given θ), is a straightforward modification of Theorem 1.

Theorem 2. *The following statements on $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ are equivalent.*
[A] *The set of observations is θ-CV rationalizable with convex cost functions, where $\theta \gg 0$.*
[B] *There exists a set of positive real numbers, $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$, that satisfy the generalized common ratio property, i.e.,*

$$\frac{P_t - \delta_{1,t}}{\theta_1 Q_{1,t}} = \frac{P_t - \delta_{2,t}}{\theta_2 Q_{2,t}} = \ldots = \frac{P_t - \delta_{I,t}}{\theta_I Q_{I,t}} > 0 \text{ for all } t \in \mathcal{T}, \tag{7}$$

and for each i, $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ is increasing with $Q_{i,t}$.

Proof: We first show that [A] implies [B]. Suppose that the data is rationalized with demand functions $\{\bar{P}_t\}_{t\in\mathcal{T}}$ and cost functions $\{\bar{C}_i\}_{i\in\mathcal{I}}$. At observation t, firm i's choice of $Q_{i,t}$ is optimal given the output of other firms and given its conjecture θ_i. By the first order condition, there is $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$ (the set of subgradients of $\bar{C}_i$ at $Q_{i,t}$) such that

$$\theta_i Q_{i,t}\bar{P}_t'(Q_t) + \bar{P}_t(Q_t) - \delta_{i,t} = \theta_i Q_{i,t}\bar{P}_t'(Q_t) + P_t - \delta_{i,t} = 0.$$

Re-arranging this equation, we obtain $-\bar{P}_t'(Q_t) = (P_t - \delta_{i,t})/\theta_i Q_{i,t}$ for all i. This gives us equation (7), which is obviously a more general version of the common ratio property. Since $\bar{C}_i$ is convex, $\delta_{i,t}$ must increase with $Q_{i,t}$.

To proof that [B] implies [A] we need only mimic the two-step procedure used in the proof of Theorem 1. Lemma 2 guarantees that firm i has a convex and well-behaved cost function $\bar{C}_i$ such that $\delta_{i,t} \in \bar{C}_i(Q_{i,t})$. A modified version of Lemma 1 is then needed to show that $Q_{i,t}$ is firm i's optimal choice at observation t, given it's conjecture θ_i. This involves constructing the right demand function. As in the proof of Lemma 1, we use the linear function $\bar{P}_t(Q) = a_t - b_t Q$, where $b_t = [P_t - \delta_{i,t}]/\theta_i Q_{i,t}$. This is well-defined because of the modified common ratio property (7). To check that $Q_{i,t}$ is optimal with this demand function is straightforward: we make an argument analogous to that used in the proof of Lemma 1. QED

One important thing to notice in Theorem 2 is the following: *if condition (7) is satisfied by θ then it is satisfied by $\lambda\theta$ for any $\lambda > 0$.* This means that θ can only be tested up to scalar multiples and testing for the absolute level of market power is impossible in our context; for example, the data set from a duopoly is $(1,1)$-CV rationalizable if and only if it is $(10,10)$-CV rationalizable. However – and this is crucial for our purposes – relative market power as measured by θ_i *is* testable; potentially, a set of observations could be consistent with, say, $\theta = (1,1)$ but not $\theta = (1,2)$.

Put another way, our minimal assumptions on costs and demand means that we could not test specifically the hypothesis that, for some firm i, $\theta_i = 1$. However, this does not mean that we could not test the Cournot model, because we could still test the weaker hypothesis that θ_i is the same across firms. The test of the Cournot model we developed in Theorem 1 can be interpreted as a test of the *symmetry* of market interaction as measured by θ_i (for all $i \in \mathcal{I}$). When a data set passes that test, it is consistent with the Cournot hypothesis, but it is also consistent with the θ-CV hypothesis, where $\theta = (\lambda, \lambda, ..., \lambda)$ for any $\lambda > 0$; in that sense, the conclusion is weak. On the other hand, when a data set fails that test, it is a strong result because all levels of symmetric market power have been excluded.

Our observations here are broadly consistent with the results of Bresnahan (1982) and Lau (1982). These authors show that the identification of θ requires sufficiently rich variation in (and information on) demand behavior across observations; in contrast, our setup requires no information on the determinants of demand. It is not hard to see that, for a modeler who is interested in narrowing down the value of θ, the introduction of more information on demand behavior will help. From the proof of Theorem 2, we know that the inverse demand function constructed to rationalize the data has slope $(P_t - \delta_{i,t})/\theta_i Q_{i,t}$. A proportionate reduction in the θ_is does not upset condition (7), but the slope of the rationalizing inverse demand function provided in the proof will decrease (i.e., the demand curve becomes steeper). This suggests that if we have information on the demand curve that bounds the elasticity of demand within some range, then θ will no longer be indeterminate up to scalar multiples. This is easily illustrated with an example.

Example 3. Consider a duopoly with firms i and j where
(i) at observation t, $P_t = 10$, $Q_{i,t} = 5/3$ and $Q_{j,t} = 5/3$; and
(ii) at observation t', $P_{t'} = 4$, $Q_{i,t'} = 2$ and $Q_{j,t'} = 5/3$.
In addition, suppose the modeler knows that $d\bar{P}_t/dq \geq -3$; loosely speaking, he knows of a bound on how quickly price falls with increased output at t. With this additional condition, we claim that the observations are compatible with $\theta = (3,3)$ but not with $\theta = (1,1)$.

Indeed, applying Theorem 2, compatibility with $\theta = (3,3)$ is confirmed if we could find $\delta_{i,t}, \delta_{i,t'}$, $\delta_{j,t}$ and $\delta_{j,t'}$ that solves

$$\frac{10 - \delta_{i,t}}{5} = \frac{10 - \delta_{j,t}}{5} \quad \text{and} \quad \frac{4 - \delta_{i,t'}}{6} = \frac{4 - \delta_{j,t'}}{5}.$$

In addition, because firm i's output is higher at t' than at t, we also require $\delta_{i,t} \leq \delta_{i,t'}$. It is straightforward to check that these conditions are met if $\delta_{i,t} = 3$, $\delta_{i,t'} = 3$, $\delta_{j,t} = 3$ and $\delta_{j,t'} = 19/6$. In this case, the rationalizing inverse demand function $\bar{P}_t$ can be chosen to satisfy $d\bar{P}_t/dq = -(10-3)/5 = -7/5$, which is greater than -3.

Suppose, contrary to our claim, that the data set is Cournot rationalizable with a rationalizing demand $\bar{P}_t$ satisfying $d\bar{P}_t/dq \geq -3$. In that case, the first order condition of firm i gives

$$\frac{10 - m_{i,t}}{5/3} = -\frac{d\bar{P}_t}{dq} \leq 3,$$

where $m_{i,t}$ is a subgradient of firm i's cost function at output $Q_{i,t} = 5/3$. Therefore, $m_{i,t} \geq 5$, which means that the marginal cost at $Q_{i,t'} = 2$ must be at least 5 since firm i's cost function is convex. However, the price at t' is just 4, so there is a contradiction.

More generally, it is clear that with this lower bound on the slope of the inverse demand function, there is λ^* such that the observations are (λ, λ)-CV rationalizable if $\lambda > \lambda^*$ and not (λ, λ)-CV rationalizable if $\lambda < \lambda^*$. It is worth comparing this example with Lau (1982), who showed that the identification of $\theta = (\lambda, \lambda)$ requires that demand be drawn from at least a two-parameter family. Our bound on the slope of demand does not permit identification as such, but it is enough to identify a range of values of λ that is consistent with the data; in certain situations, this coarser information may be all that (say) an industry regulator is interested in.

We now outline a test for collusion that encompasses Example 3. The modeler assumes that the data set is generated by demand varying across observations, while cost functions are fixed. In addition, the inverse demand functions are drawn from a specified family $\mathcal{P}$. Besides observing P_t and $(Q_{i,t})_{t\in\mathcal{T}}$ at each t, the observer also observes an n-vector of parameters $z \in Z \subset R^n$ that has a known impact on the elasticity of demand. More precisely, we associate to each observation $(q, p, z) \in R_{++} \times R_{++} \times Z$ a set $S(q, p, z) \subset (-\infty, 0)$. Having observed (q, p, z), the observer assumes that the demand can be any function $\bar{P}$ drawn from $\mathcal{P}$, satisfying $\bar{P}(q) = p$ and with $\bar{P}'(q) \in S(q, p, z)$.

We assume that $\mathcal{P}$ is a *flexible family* of inverse demand functions, by which we mean that the following holds: (i) every element in $\mathcal{P}$ is downward sloping and log-concave and (ii) given any observation $(q, p) \in R^2_{++}$ and any negative slope η, there is an element in $\mathcal{P}$ passing through (q, p) and with slope η at that point. One example is the family of linear inverse demand functions, with a typical element of the form $\bar{P}(q) = a - bq$ (where a and b are positive numbers). Another example is the family of exponential inverse demand functions, where $\bar{P}(q) = Ae^{-Bq}$ and A and B are positive scalars. Note that both families are also log-concave over that part of the domain where the price is positive.

Theorem 3. *Given the correspondence I and a flexible family $\mathcal{P}$, the following statements on $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, z_t]\}_{t\in\mathcal{T}}$ (with $P_t > 0$ for all t) are equivalent.*
[A] *The set of observations is θ-CV rationalizable (for $\theta \gg 0$) with convex cost functions and with the rationalizing inverse demand functions $\bar{P}_t$ drawn from $\mathcal{P}$ and satisfying $\bar{P}_t'(Q_t) \in S(Q_t, P_t, z_t)$.*
[B] *There exists a set of positive real numbers $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ that obey the generalized common ratio property (7) and, for each i, $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ is increasing with $Q_{i,t}$. Furthermore, $[P_t - \delta_{1,t}]/\theta_1 Q_{1,t} \in S(Q_t, P_t, z_t)$ for all $t \in \mathcal{T}$.*

We shall omit the proof of this result since it can be done by an obvious modification of the proofs given in Theorems 1 and 2. One important thing to note is that in proving that [B] implies [A], the flexibility of $\mathcal{P}$ guarantees that there is a demand function $\bar{P}_t$ in $\mathcal{P}$ such that $\bar{P}_t(Q_t) = P_t$ and $\bar{P}_t'(Q_t) = [P_t - \delta_{i,t}]/\theta_i Q_{i,t}$. The first order condition for each firm i then implies global optimality because the log-concavity of $\bar{P}_t$ means that each firm's profit function is quasiconcave in its output (see Vives (Section 6.2, 1999)).

4. Cournot Rationalizability in a multi-product oligopoly

In this section, we move away from the single-product setting and consider a market consisting of I firms, with each firm producing up to m goods. The production costs and demand for these goods are possibly inter-related (see, for example, Brander and Eaton (1984) and Bulow et al. (1985)). As in the single-good case, we consider a scenario where an observer makes T observations of this market, with each observation consisting of the prices of the m goods, and the output of each good by each firm. Formally, each observation t consists of the price vector $P_t = (P_t^k)_{k \in M}$ (where $M = \{1, 2, ..., m\}$ is the set of goods) and the output vector of each firm; for firm i, this is $Q_{i,t} = (Q_{i,t}^k)_{k \in M}$. So the set of observations may be denoted as $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$. We require that this set of observations satisfy $Q_{i,t} > 0$ for every firm i and at every observation t, and also that $\sum_{i \in \mathcal{I}} Q_{i,t} \gg 0$. In other woods, every firm is always producing something (though a firm need not produce every one of the m goods) and strictly positive amounts of each good is produced at all observations.

The set of observations $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ is Cournot rationalizable if each observation can be explained as a Cournot equilibrium arising from a different market demand function, keeping the cost function of each firm fixed across observations. We impose some regularity conditions on the demand and cost functions. Generalizing our earlier definition (for the single-product case), a cost function of firm i is a function $\bar{C}_i : \mathbb{R}_+^m \to \mathbb{R}$ such that $\bar{C}_i(0) = 0$, $\bar{C}_i$ is nondecreasing with respect to $q \in \mathbb{R}_+^m$, and $\bar{C}_i$ is strictly increasing along rays.[8] We require the market inverse demand function $\bar{P}_t : \mathbb{R}_+^m \to \mathbb{R}^m$ (for each t) to obey the *law of demand*; by this we mean that $\bar{P}_t$ is differentiable with a negative definite derivative matrix $\partial \bar{P}_t$. In particular, this implies that the diagonal terms of $\partial \bar{P}_t$ (the own-price derivative for any good) are negative numbers, but negative definiteness is a stronger property. This generalization of the downward-sloping property is not the only one possible, but it is intuitive, convenient for our purposes, and has been extensively

[8] Since the cost function need not be additive across goods, synergies in production are allowed.

studied.[9] It implies that for $\tilde{q} \neq \tilde{q}'$, we have $(\tilde{q} - \tilde{q}') \cdot (\bar{P}_t(\tilde{q}) - \bar{P}_t(\tilde{q}')) < 0$.[10]

The set of observations $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ from an m-product market is said to be *Cournot rationalizable* if there exists cost functions $\bar{C}_i$ for each firm i, and demand functions $\bar{P}_t$ obeying the law of demand for each observation t such that
(i) $\bar{P}_t(Q_t) = P_t$; and
(ii) $Q_{i,t} \in \text{argmax}_{q_i \geq 0} \left\{ \sum_{k=1}^m q_i^k \bar{P}_t^k (q_i + \sum_{j \neq i} Q_{j,t}) - \bar{C}_i(q_i) \right\}$.
Condition (i) says that the inverse demand function must agree with the observed data at each t. Condition (ii) says that, at each observation t, firm i's observed output, $Q_{i,t} = (Q_{i,t}^k)_{k \in M}$, maximizes its profit given the output of the other firms.

Theorem 4 below is the main result of this section and is the multi-product generalization of Theorem 1. It gives necessary and sufficient conditions on a data set to be Cournot rationalizable with convex cost functions.

THEOREM 4. *The following statements on the set of observations $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ are equivalent.*
[A]. *The set of observations is Cournot rationalizable with with convex cost functions.*
[B]. *There exists real numbers $\lambda_t^{\ell,k}$, nonnegative numbers $\delta_{i,t}^k$ and positive numbers $C_{i,t}$ such that, for all ℓ and $k \in M$, t and $t' \in \mathcal{T}$, and $i \in \mathcal{I}$, the following holds:*
(i) the $M \times M$ matrix $\Lambda_t = \left[\lambda_t^{\ell,k} \right]$ is positive definite;
(ii) $\delta_{i,t}^k - P_t + \sum_{\ell=1}^m \lambda_t^{\ell,k} Q_{i,t}^\ell \geq 0$ and $\left(\delta_{i,t}^k - P_t + \sum_{\ell=1}^m \lambda_t^{\ell,k} Q_{i,t}^\ell \right) Q_{i,t}^k = 0$;
(iii) $C_{i,t'} \geq C_{i,t} + \sum_{k=1}^m \delta_{i,t}^k (Q_{i,t'}^k - Q_{i,t}^k)$; and
(iv) $0 \geq C_{i,t} - \sum_{k=1}^m \delta_{i,t}^k Q_{i,t}^k$.

As in the proof of Theorem 1, we prove this result via two lemmas, with the first analogous to Lemma 1 and the second to Lemma 2.

LEMMA 3. *Suppose that, at some observation t, there are real numbers $\lambda_t^{\ell,k}$ and nonnegative numbers $\delta_{i,t}^k$ such that, for all ℓ and $k \in M$ and $i \in \mathcal{I}$, conditions (i) and (ii) in Theorem 4 are satisfied. In addition, suppose that there are convex cost functions $\bar{C}_i$ with $(\delta_{i,t}^k)_{k \in M} \in \partial \bar{C}_i(Q_{i,t})$. Then there exists an inverse demand function $\bar{P}_t$ obeying the law of*

[9]For the use of this condition in the context of multi-product oligopolies, see Vives (1999). The microfoundations of this property has also been extensively studied; see Quah (2003) and also the survey of Jerison and Quah (2008). The literature tends to consider demand as a function of price, rather than the inverse demand function considered here. However, the two cases are equivalent: if $\partial^2 \bar{P}_t(\tilde{Q})$ is negative definite, then it is locally invertible, and its inverse (i.e. the demand function) $\bar{D}_t$ has a negative definite matrix at the price vector $\bar{P}_t(\tilde{Q})$.

[10]Indeed, the two properties are effectively equivalent. To be precise, $(\tilde{q} - \tilde{q}') \cdot (\bar{P}_t(\tilde{q}) - \bar{P}_t(\tilde{q}')) \leq 0$, for all q and q' in a convex and open set $\mathcal{O}$ if and only if $\partial^2 \bar{P}_t(q)$ is negative semidefinite for all $q \in \mathcal{O}$.

demand such that $\bar{P}_t(Q_t) = P_t$ and, with each firm i having the cost function $\bar{C}_i$, $\{Q_{i,t}\}_{i\in\mathcal{I}}$ constitutes a Cournot equilibrium.

Proof: We define the inverse demand function for good k by $\bar{P}^k_t(q) = a^k_t - \sum_{\ell=1}^m \lambda^{k,\ell}_t q^\ell$ with a^k_t chosen such that $\bar{P}^k_t(Q_t) = P^k_t$. Firm i's profit at observation t, given that firm $j \neq i$ is producing $Q_{j,t}$ is $\Pi_{i,t}(q_i) = \bar{R}_{i,t}(q_i) - \bar{C}_i(q_i)$, where the revenue function $\bar{R}_{i,t}$ has the form

$$\bar{R}_{i,t}(q_i) = \sum_{\ell=1}^m q^\ell_i \bar{P}^\ell_t \left(q^\ell_i + \sum_{j\neq i} Q^\ell_{j,t} \right).$$

Note that

$$\frac{\partial \bar{R}_{i,t}}{\partial q^k_i}(Q_{i,t}) = \bar{P}^k_t(Q_t) + \sum_{\ell=1}^m \frac{\partial \bar{P}^\ell_t}{\partial q^k}(Q_t) Q^\ell_{i,t} = P^k_t - \sum_{\ell=1}^m \lambda^{\ell,k}_t Q^\ell_{i,t}. \tag{8}$$

Since $(\delta^k_{i,t})_{k\in M} \in \partial \bar{C}_i(Q_{i,t})$, condition (ii) gives the Kuhn-Tucker conditions for profit maximization. These conditions are sufficient to guarantee that firm i's choice is optimal if $\Pi_{i,t}$, is concave in q_i. Given that $\bar{C}_i$ is a convex function, it suffices to check that the $\bar{R}_{i,t}$ is concave in q_i. It is straightforward to verify that, for all q_i, the Hessian $\partial^2 \bar{R}_{i,t}(q_i) = -\Lambda^T_t - \Lambda_t$. Condition (i) guarantees that this matrix is negative definite, so we conclude that $\bar{R}_{i,t}$ is concave. QED

LEMMA 4. *Suppose that for some firm i, there are nonnegative numbers $\delta^k_{i,t}$ and positive numbers $C_{i,t}$ such that, for all $k \in M$ and t and $t' \in \mathcal{T}$, and $i \in \mathcal{I}$, condition (iii) and (iv) in Theorem 4 are satisfied. Then there exists a convex cost function such that $(\delta^k_{i,t})_{k\in M} \in \partial \bar{C}_i(Q_{i,t})$.*

Proof: Let $d = -\max_{t\in\mathcal{T}}\{C_{i,t} - \sum_{k=1}^m \delta^k_{i,t} Q^k_{i,t}\}$; by (iv), $d \geq 0$. Given this, $\hat{C}_{i,t} = C_{i,t} + d$ is a (strictly) positive number since $C_{i,t}$ is a positive number. Define the function $\tilde{C}_i$ by

$$\tilde{C}_i(q) = \max_{t\in\mathcal{T}}\{\hat{C}_{i,t} + \sum_{k=1}^m \delta^k_{i,t}(q^k - Q^k_{i,t})\}. \tag{9}$$

The function $\tilde{C}_i$ has all but one of the conditions we require on the cost function. First, notice that our choice of d guarantees that $\tilde{C}_i(0) = 0$. Since $\delta^k_{i,t} \geq 0$, the function $\tilde{C}_i$ is nondecreasing and since it is the upper envelope of linear functions, $\tilde{C}_i$ is a convex function. Condition (iii) implies that $\tilde{C}_i(Q_{i,t}) = \hat{C}_{i,t} > 0$ since

$$\hat{C}_{i,t} \geq \hat{C}_{i,s} + \sum_{k=1}^m \delta^k_{i,s}(Q^k_{i,t} - Q^k_{i,s}) \text{ for all } s \in \mathcal{T}.$$

Therefore, $(\delta^k_{i,t})_{k\in M} \in \partial \tilde{C}_i(Q_{i,t})$.

However, $\tilde{C}_i$ may not be strictly increasing along rays. To guarantee this property we modify the function $\tilde{C}_i$ in the following way. Choose a vector $\bar{\epsilon} = (\epsilon, \epsilon, ..., \epsilon)$ with $\epsilon > 0$ and sufficiently small so that $C_{i,t} > \bar{\epsilon} \cdot Q_{i,t}$ for all t. Define the function $\bar{C}_i$ by $\bar{C}_i(q) = \max\{\tilde{C}_i(q), \bar{\epsilon} \cdot q\}$; $\bar{C}_i$ is a convex and nondecreasing function, with $\bar{C}_i(0) = 0$ and $\bar{C}_i(Q_{i,t}) = \hat{C}_{i,t}$. Locally at $Q_{i,t}$, $\bar{C}_i$ and $\tilde{C}_i$ are identical, so $(\delta^k_{i,t})_{k\in M} \in \partial\bar{C}_i(Q_{i,t})$. In addition, $\bar{C}_i$ is strictly increasing along rays. Suppose, to the contrary, that $\bar{C}_i$ is locally constant along the ray through the point $q = \underline{q}$. In that case, there exists $\bar{s} \in \mathcal{T}$ such that $\hat{C}_{i,\bar{s}} + \sum_{k=1}^m \delta^k_{i,\bar{s}}(\lambda\underline{q}^k - Q^k_{i,\bar{s}})$ is constant and positive for all values of λ (and thus including $\lambda = 0$), which is not possible since $\bar{C}_i(0) = 0$. QED

Proof of Theorem 4: Suppose that [A] holds, so the data could be rationalized by inverse demand functions $\bar{P}^k_t$, for $k \in M$ and $t \in \mathcal{T}$ and cost functions $\bar{C}_i$. We set

$$\lambda^{\ell,k}_t = -\frac{\partial \bar{P}^\ell_t}{\partial q^k}(Q_t).$$

Since $(\bar{P}^k_t)_{k\in M}$ obeys the law of demand, Λ_t is positive definite as required by (i).

At observation t, the marginal revenue for firm i as it varies the output of good k is $P^k_t - \sum_{\ell=1}^m \lambda^{\ell,k}_t Q^\ell_{i,t}$ (see (8)). Since $Q_{i,t}$ is optimal for firm i, there exists a vector $(\delta^k_{i,t})_{k\in M}$ in $\partial\bar{C}_i(Q_{i,t})$ such that $\delta^k_{i,t} \geq P^k_t - \sum_{\ell=1}^m \lambda^{\ell,k}_t Q^\ell_{i,t}$ and with equality whenever $Q^k_{i,t} > 0$ for good k, so that (ii) holds. Since $\bar{C}_i$ is nondecreasing, we may choose the subgradient $(\delta^k_{i,t})_{k\in M}$ to be a nonnegative vector.

Given that $\bar{C}_i$ is strictly increasing along rays and $Q_{i,t} > 0$, we have $\bar{C}_i(Q_{i,t}) > 0$ for all (i, t). Set $C_{i,t} = \bar{C}_i(Q_{i,t})$; since $\bar{C}_i$ is convex and $(\delta^k_{i,t})_{k\in M}$ is a subgradient, (iii) holds. Finally, (iv) holds since $\bar{C}_i$ is convex and $\bar{C}_i(0) = 0$. This completes our proof that [A] implies [B].

The fact that [B] implies [A] follows immmediately from Lemmas 3 and 4. QED

Theorem 4 would be meaningless if in fact there are no observable restrictions in a multi-product Cournot game. To remove this possibility, we now provide an example of a data set that is *not* compatible with Cournot interaction.

Example 4. Consider an industry with two goods, 1 and 2. Observations taken from two firms in this industry are as follows:

(i) at observation t, $P^1_t = 10$, $Q^1_{i,t} = 13$, $Q^2_{i,t} = 12$, $Q^1_{j,t} = 4$, $Q^2_{j,t} = 6$.

(ii) at observation t', $P^1_{t'} = 1$, $P^2_{t'} = 1$, $Q^1_{j,t} = 8$ and $Q^2_{j,t} = 8$.

Suppose that the observations at t constitutes a Cournot equilibrium. In that case, the first order condition for firm i says that there is $(\delta^1_{i,t}, \delta^2_{i,t}) \in \partial \bar{C}_i(Q_{i,t})$ such that

$$\bar{P}^1_t(Q_t) + Q^1_{i,t}\frac{\partial \bar{P}^1_t}{\partial q_1} + Q^2_{i,t}\frac{\partial \bar{P}^2_t}{\partial q_1} - \delta^1_{i,t} = 0.$$

Similarly, the first order condition for firm j says that there is $(\delta^1_{j,t}, \delta^2_{j,t}) \in \partial \bar{C}_j(Q_{j,t})$ such that

$$\bar{P}^1_t(Q_t) + Q^1_{j,t}\frac{\partial \bar{P}^1_t}{\partial q_1} + Q^2_{j,t}\frac{\partial \bar{P}^2_t}{\partial q_1} - \delta^1_{j,t} = 0.$$

Multiplying the first equation by $Q^2_{j,t}$ and the second equation by $Q^2_{i,t}$ and taking the difference between them, we obtain

$$(Q^2_{j,t} - Q^2_{i,t})\bar{P}^1_t(Q_t) + \left[Q^2_{j,t}Q^1_{i,t} - Q^2_{i,t}Q^1_{j,t}\right]\frac{\partial \bar{P}^1_t}{\partial q_1} - Q^2_{j,t}\delta^1_{i,t} + Q^2_{i,t}\delta^1_{j,t} = 0. \tag{10}$$

The significance of the numbers chosen for observation t is that they guarantee that $Q^2_{j,t} - Q^2_{i,t} < 0$ and $Q^2_{j,t}Q^1_{i,t} - Q^2_{i,t}Q^1_{j,t} > 0$. Note $\delta^1_{i,t} \geq 0$ since firm i's cost function is nondecreasing and $\partial \bar{P}^1_t/\partial q_1 < 0$ because of the law of demand; therefore the second and third terms on the left of equation (10) are both negative. Re-arranging that equation, we obtain

$$\delta^1_{j,t} \geq \frac{(Q^2_{i,t} - Q^2_{j,t})}{Q^2_{i,t}}\bar{P}^1_t(Q_t) = \frac{6}{12} \cdot 10 = 5. \tag{11}$$

In short, observation t provides us a with a lower bound on the marginal cost of firm j at its observed output of $(4, 6)$.

At observation t', firm j's output is $(8, 8)$. The marginal cost of increasing output from $(4, 6)$ to $(8, 8)$ is no smaller than the marginal cost of increasing output from $(4, 6)$ to $(8, 6)$, which is in turn bounded below by $5 \times 4 = 20$ (because of (11) and the convexity of $\bar{C}_j$). So the total cost of producing $(8, 8)$ is at least 20 but the total revenue of firm i at observation t' is just 16: firm i is better off choosing $(0, 0)$ at observation t'. We conclude that observations t and t' cannot both be Cournot outcomes.

The multi-product setting of Theorem 4 raises a number of issues not present in the single product setting of Theorem 1. We consider them in turn.

Like Theorem 1, Theorem 4 establishes an equivalence between Cournot rationalizability and the solution to a programming problem. However, the program in statement [B] of Theorem 4 is not a linear program, because it requires checking that the matrix Λ is positive definite (condition (i) in statement [B]). This condition is required for the precise reason that we require the market demand function to obey the law of demand. It is possible to replace the law of demand with a stronger condition that is easier to check. For example, we could require the rationalizing inverse demand function $\bar{P}_t$ to obey *diagonal dominance with uniform weights*; by this, we mean that

$$2\frac{\partial \bar{P}_t^k}{\partial q_k}(q) + \sum_{\ell \neq k} \left| \frac{\partial \bar{P}_t^k}{\partial q_\ell}(q) + \frac{\partial \bar{P}_t^\ell}{\partial q_k}(q) \right| < 0 \text{ for all } q \text{ and for all } k \in M.$$

This intuitive condition says that own-price effects are larger than the sum of all cross-price effects. If we impose this condition on the rationalizing demand system, then the corresponding condition on $\delta_{i,t}^k$ (in place of condition (i) in [B]) is the following:

$$-2\lambda_t^{k,k} + \sum_{\ell \neq k} \left| \lambda_t^{\ell,k} + \lambda_t^{k,\ell} \right| < 0 \text{ for all } k \in M;^{11}$$

note that this can condition can be equivalently stated as a set of linear conditions.

In certain contexts, the modeler may have specific information on cross price effects which he would like to impose as conditions on the rationalizing demand system, on top of those required by the law of demand or diagonal dominance. For example, it is possible to interpret the different goods in this model as the same good sold in several distinct and isolated markets; in other words, this multi-product oligopoly is an instance of third degree price discrimination, with the same firms interacting in several markets. In that case, it may be reasonable to require all cross price effects to equal zero, i.e., $\partial \bar{P}_t^k / \partial q^\ell = 0$ for all $k \neq \ell$. Correspondingly, one would have to impose the condition $\lambda_t^{\ell,k} = 0$ for all t and whenever $\ell \neq k$, in addition to the ones listed in statement [B] (of Theorem 4).

Similarly, the modeler may believe that the m goods are substitutes ($\partial \bar{P}_t^k / \partial q^\ell \leq 0$ for all ℓ and k) or complements ($\partial \bar{P}_t^k / \partial q^\ell \geq 0$ for all $\ell \neq k$). The corresponding conditions are $\lambda_t^{\ell,k} \leq 0$ for all ℓ and k and $\lambda_t^{\ell,k} \geq 0$ for all $\ell \neq k$ respectively.

If we impose the condition that all m goods are substitutes of each other then Cournot rationalizability requires that all observed prices (P_t^k for all t and k) be non-negative. Indeed, if $P_t^{\bar{k}} < 0$ then any firm that is producing good $\bar{k}$ is strictly better off if it reduces its output of $\bar{k}$ (which strictly increases revenue and at least weakly lowers costs). In the case when the goods are not necessarily substitutes, the model allows for the possibility that some observed prices are negative. In other words, firms can optimally *pay* for a good to be consumed in order that it may raise the demand for some other good. It is not hard to construct examples displaying this phenomenon.

[11] This property guarantees the positive definiteness of the symmetric matrix $\Lambda + \Lambda^T$, which is equivalent to the positive definiteness of Λ (see Mas-Colell et al. (Appendix M.D, 1995) for more on diagonal dominance).

5. Convincing Cournot Rationalizability

In all our results so far, we have studied Cournot rationalizability when cost functions display increasing marginal costs. This assumption on cost functions is of course ubiquitous in both theoretical and empirical work; its great advantage is to ensure that the first order conditions are not just necessary, but also sufficient, for optimality. Nonetheless, in the context of oligopoly games, where increasing returns to scale may be present, it is useful to have a test for the Cournot hypothesis that is not necessarily linked to firms having increasing marginal costs.

In the one-good context, we could ask what conditions are needed for $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ to be Cournot rationalizable if we allow cost curves to be non-convex.[12] The following result says that Cournot competition imposes no restrictions on any generic set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$; by *generic* we mean that, for all i, $Q_{i,t} \neq Q_{i,t'}$ whenever $t \neq t'$.

Proposition 2. *Any generic set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is Cournot rationalizable with firms having C^2 cost functions.*

To prove this result, and also to see how we could work around its seemingly negative implication, it is useful to first consider a scenario where the observer has more information at his disposal.

Suppose that, in addition to price and output, the observer also observes the total cost incurred by each firm. Formally, the set of observations is $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$, where $C_{i,t} > 0$ is the total cost incurred by firm i at output $Q_{i,t} > 0$; we require $C_{i,t} = C_{i,t'}$ if $Q_{i,t} = Q_{i,t'}$. We say that this set is Cournot rationalizable if there are downward sloping (hence differentiable) inverse demand functions $\bar{P}_t$ (for all $t \in \mathcal{T}$) and cost functions $\bar{C}_i$ (for each firm $i \in \mathcal{I}$) such that $\bar{P}_t(Q_t) = P_t$, $\bar{C}_i(Q_{i,t}) = C_{i,t}$, and $(Q_{i,t})_{i\in\mathcal{I}}$ is a Cournot equilibrium when demand is $\bar{P}_t$.

What restrictions does Cournot rationalizability impose on this data set? To answer this question, we first need to introduce some notation. For each i and t, define the set

$$L_i(t) = \{t' \in \mathcal{T} : Q_{i,t'} < Q_{i,t}\} \cup \{0\}.$$

$L_i(t)$ consists of those observations where firm i's output is strictly lower than Q_t, as well as a fictitious observation 0, for which $Q_0 = 0$ and $C_{i,0} = 0$. We denote the observation where firm i's output is the lowest by t_i^*. It follows that $L_i(t_i^*) = \{0\}$ whilst, for any $t \neq t_i^*$, $L_i(t)$

[12] Recall though, that our definition of cost curves require that they be continuous, strictly increasing, and has no cost at output zero.

will contain t_i^*, 0, and possibly other observations. We denote $l_i(t) = \text{argmax}_{t' \in L_i(t)} Q_{i,t'}$; that is, $l_i(t)$ is the set of observations corresponding to the highest output level strictly below $Q_{i,t}$.[13] In a similar fashion, the observation with the highest output level for firm i is denoted by t_i^{**}. For $t \neq t^{**}$, the set of observations with outputs strictly higher than t is denoted by $U_i(t)$, with $u_i(t) = \text{argmin}_{t' \in U_i(t)} Q_{i,t'}$, so $u_i(t)$ is the observation with the lowest output level above $Q_{i,t}$.

For any t in $\mathcal{T}$, define $\mathrm{d}Q_{i,t} = Q_{i,t} - Q_{i,l_i(t)}$ and $\mathrm{d}C_{i,t} = C_{i,t} - C_{i,l(t)}$. In words, $\mathrm{d}C_{i,t}$ is the extra cost incurred by firm i when it increases its output from $Q_{i,l(t)}$ to $Q_{i,t}$. We denote the *average marginal cost* over that output range by $M_{i,t} = \mathrm{d}C_{i,t}/\mathrm{d}Q_{i,t}$.

We say that $\{C_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ obeys the *discrete marginal property* if for every i and t, the following holds:

$$C_{i,t} - C_{i,t'} < P_t(Q_{i,t} - Q_{i,t'}) \;\text{ for } t' \in L_i(t). \tag{12}$$

For any $t' \in L_i(t)$, let $\mathbb{Q}_i(t', t)$ denote the set consisting of $Q_{i,t}$ and those observed output levels of firm i strictly between $Q_{i,t}$ and $Q_{i,t'}$. Formally,

$$\mathbb{Q}_i(t', t) = \{Q_{i,s} : s \in (L_i(t) \cup \{t\}) \setminus (L_i(t') \cup \{t'\})\,\}.$$

Since $C_{i,t} - C_{i,t'} = \sum_{Q_{i,s}\in\mathbb{Q}_i(t',t)} M_{i,s}(Q_{i,s} - Q_{i,l(s)})$ the discrete marginal property may also be written as

$$\sum_{Q_{i,s}\in\mathbb{Q}_i(t',t)} M_{i,s}(Q_{i,s} - Q_{i,l(s)}) < P_t(Q_{i,t} - Q_{i,t'}) \;\text{ for } t' \in L_i(t). \tag{13}$$

We claim that this property is *necessary* for Cournot rationalizability. Indeed, notice that instead of producing at $Q_{i,t}$, firm i could have chosen to produce at $Q_{i,t'}$ for some $t' \in L_i(t)$ (that is, at a lower level of output). Given that $Q_{i,t}$ was chosen, the additional cost incurred, which is $C_{i,t} - C_{i,t'}$ must be less than the additional revenue gained, and the latter is bounded by $P_t(Q_{i,t} - Q_{i,t'})$ (because the demand curve is downward sloping). The next result says that the discrete marginal property is both necessary and sufficient for Cournot rationalizability.

THEOREM 5. *A generic set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is Cournot rationalizable with C^2 cost functions if and only if it obeys the discrete marginal property.*

The proof of this result requires the following lemma.

[13]In particular, $l_i(t_i^*) = \{0\}$. When the data set is generic, $l_i(t)$ is always singleton; otherwise it could have more than one element.

Lemma 5. *Let* $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ *be a generic set of observations obeying the discrete marginal property and suppose that the positive numbers* $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ *satisfy* $0 < \delta_{i,t} < P_t$*, for all* (i,t)*, with* $\delta_{i,t} = \delta_{i,t'}$ *whenever* $Q_{i,t} = Q_{i,t'}$*. Then, there are* C^2 *cost functions* $\bar{C}_i : \mathbb{R}_+ \to \mathbb{R}$ *such that*
(i) $\bar{C}_i(Q_{i,t}) = C_{i,t}$*;*
(ii) $\bar{C}_i'(Q_{i,t}) = \delta_{i,t}$*; and*
(iii) for all q_i *in* $[0, Q_{i,t})$*,*

$$P_t q_i - \bar{C}_i(q_i) < P_t Q_{i,t} - \bar{C}_i(Q_{i,t}). \tag{14}$$

Proof: Note that the inequality (14) may be re-written as

$$\bar{C}_i(q_i) > P_t(q_i - Q_{i,t}) + \bar{C}_i(Q_{i,t}). \tag{15}$$

The function $f_t(q_i) = P_t(q_i - Q_{i,t}) + C_{i,t}$, for q_i in $[0, Q_{i,t})$, is represented by a line with slope P_t passing through the point $(Q_{i,t}, C_{i,t})$ – see Figure 1. Condition M guarantees that for t' in $L_i(t)$, $(Q_{i,t'}, C_{i,t'})$ lies *above* the line f_t. We require a cost function that satisfies (15). One such function is the one given by the linear interpolation of all the points $(Q_{i,t}, C_{i,t})$, since its graph stays above every one of the lines representing the functions f_t. This cost function can in turn be replaced by a C^2 function where the derivative at $Q_{i,t}$ is $\delta_{i,t}$, since $\delta_{i,t} < P_t$ and the latter is the slope of f_t. QED

This lemma says that there is a cost function for firm i that (i) agrees with the observed values of firm costs, (ii) has marginal cost agreeing with specified values at the observed output levels, and (iii) guarantees that $q = Q_{i,t}$ is the optimal output level for firm i if the inverse demand function at t is $\tilde{P}_t(q) = P_t$ for $q \le Q_t$ and $\tilde{P}_t(q) = 0$ for $q > Q_t$. So we have almost proved Theorem 5, and we fall a bit short only because the rationalizing demand function $\tilde{P}_t$ we just provided is not differentiable or downward sloping in our sense. However, as we show in the next result, it is always possible to replace $\tilde{P}_t$ with a downward sloping inverse demand function $\bar{P}_t$ that preserves the optimality of firm i's output choice.

Lemma 6. *Let* $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ *be a set of positive numbers, with* $\delta_{i,t} = \delta_{i,t'}$ *whenever* $Q_{i,t} = Q_{i,t'}$*, satisfying the common ratio property (1) and suppose that the* C^2 *cost functions* $\bar{C}_i : \mathbb{R}_+ \to \mathbb{R}$ *satisfy properties (i)-(iii) in Lemma 5. Then there are downward sloping inverse demand functions* $\bar{P}_t : \mathbb{R}_+ \to \mathbb{R}$ *such that,* $\bar{P}_t(\sum_{i\in\mathcal{I}} Q_{i,t}) = P_t$ *and, for every* i*,*

$$\text{argmax}_{q_i \ge 0} \left\{ q_i \bar{P}_t(q_i + \sum_{j\neq i} Q_{j,t}) - \bar{C}_i(q_i) \right\} = Q_{i,t}.$$

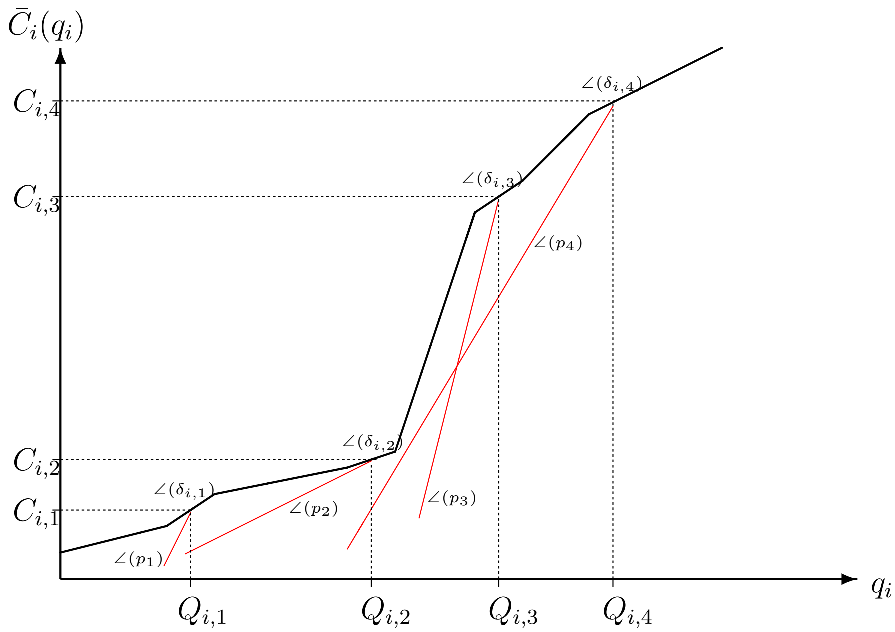

Figure 1: Construction of a Cost Function. The notation $\angle(\delta)$ is used to denote the slope (δ) at a point on the curve or of a line. The straight, thin lines represent the functions $f_t(q_i) = C_{i,t} + P_t(q_i - Q_{i,t})$. The discrete marginal property guarantees that if $Q_{i,t'} < Q_{i,t}$, then $(Q_{i,t'}, C_{i,t'})$ lies above the graph of f_t.

The proof of this lemma is deferred to the Appendix.

Proof of Theorem 5: We have already explained why the discrete marginal property is necessary for rationalizability. For sufficiency, the crucial observation to make is that the common ratio property by itself imposes no restrictions on the data set, i.e., given P_t and $\{Q_{i,t}\}_{i\in\mathcal{I}}$ there *always* exist positive numbers $\{\delta_{i,t}\}_{i\in\mathcal{I}}$ such that (1) holds. Indeed, suppose firm k produces more than any other firm at observation t, i.e., $Q_{k,t} \geq Q_{i,t}$ for all i in I. Let $\delta_{k,t}$ be any positive number smaller than P_t, and define $\beta = (P_t - \delta_{k,t})/Q_{k,t}$. Then,

$$\delta_{i,t} := P_t - \beta Q_{i,t} \geq P_t - \beta Q_{k,t} = \delta_{k,t} > 0.$$

Note also that the genericity of the data set means that the condition that $\delta_{i,t} = \delta_{i,t'}$ when $Q_{i,t} = Q_{i,t'}$ is vacuously satisfied. These observations, together with Lemmas 5 and 6, establish the sufficiency of the discrete marginal property. QED

Proof of Proposition 2: This is straightforward given Theorem 5. By that theorem, it suffices that we find an array of individual costs, $\{C_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$, that satisfies the discrete marginal property. Equivalently, we need to find $\{M_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ that obeys (13). But since the right side of that inequality is always positive and bounded away from zero for any t and t', it is clear that (13) holds if $M_{i,t}$ is sufficiently small. QED

When costs are not directly observable, Cournot rationalizabilty requires that there be $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ satisfying the common ratio property and $\{C_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ (equivalently, $\{M_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$) obeying the discrete marginal property. The former is a condition on the *infinitesimal* marginal costs of each firm while the latter is a condition on the *average* marginal costs of each firm. Both conditions in fact impose no restrictions, in the sense that, given any data set, it is *always* possible to find $\delta_{i,t}$ and $M_{i,t}$ that satisfy those conditions.

What makes Proposition 2 possible – and also what makes its seemingly negative conclusion less than persuasive – is that there need be no link between $\delta_{i,t}$ and $M_{i,t}$. In fact the proof relies crucially on the freedom to choose $M_{i,t}$ to be arbitrarily small, so that it could be considerable smaller than both $\bar{C}_i'(Q_{i,l(t)}) = \delta_{i,l(t)}$ and $\bar{C}_i'(Q_{i,t}) = \delta_{i,t}$. Since we impose no restrictions on $\bar{C}_i'$ (apart from it being a continuous function of output), this is formally permissible, but a rationalizing cost function for firm i that requires the modeler (or his audience) to believe in such a disconnection between infinitesimal and average marginal costs is not persuasive.[14]

One way of avoiding such ill-behaved marginal cost functions is to take as the marginal cost function between $Q_{i,t}$ and $Q_{i,l_i(t)}$ the linear interpolation of the hypothesized marginal costs at those two outputs, i.e, the marginal cost curve is the straight line joining $(Q_{i,l(t)}, \delta_{i,l(t)})$ and $(Q_{i,t}, \delta_{i,t})$. In that case, the average marginal cost between those two outputs is exactly $[\delta_{i,l_i(t)} + \delta_{i,t}]/2$. We wish to have more flexibility in our choice of the marginal cost function than simply taking a linear interpolation, but what we *could* require is that the rationalizing cost function's average marginal cost between those two outputs is at least $[\delta_{i,l_i(t)} + \delta_{i,t}]/2$.

Formally, a C^2 cost function $\bar{C}_i$ for firm i is said to be *convincing* or to satisfy *the convincing criterion* (given observed output $\{Q_{i,t}\}_{t\in\mathcal{T}}$) if

$$\frac{\bar{C}_i(Q_{i,t}) - \bar{C}_i(Q_{i,l_i(t)})}{Q_{i,t} - Q_{i,l(t)}} \geq \frac{1}{2}\left[\bar{C}_i'(Q_{i,l(t)}) + \bar{C}_i'(Q_{i,t})\right] \quad \text{for all } t \neq t_i^* \; . \tag{16}$$

This is illustrated in Figure 2, which depicts two marginal cost curves. The convincing criterion is violated in (a), since the area under the curve is clearly less than

$$\frac{1}{2}\left(\bar{C}_i'(Q_{i,l(t)}) + \bar{C}_i'(Q_{i,t})\right)(Q_{i,t} - Q_{i,l(t)})$$

while the criterion is satisfied in (b).

[14] Put another way, the observer is asked to believe that the marginal cost information he could surmise by observing market shares convey no information at all about costs *between* observed output levels.

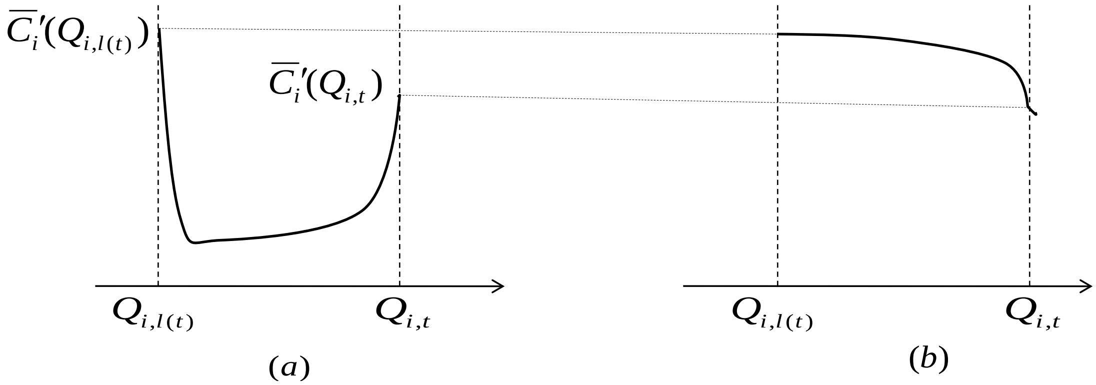

Figure 2: Cost functions that violate (a) and satisfy (b) the convincing criterion

A set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is said to be *convincingly Cournot rationalizable* if there are downward sloping demand functions $\bar{P}_t$ (for all $t \in \mathcal{T}$) with $\bar{P}_t(Q_t) = P_t$ and convincing cost functions $\bar{C}_i$ (for $i \in \mathcal{I}$) such that $(Q_{i,t})_{i\in\mathcal{I}}$ is a Cournot equilibrium at observation t. The convincing criterion is by no means the only defensible restriction that one could impose on the rationalizing cost functions; for example, 1/2 in (16) could be replaced by a lower or higher fraction.[15] However, the possibility of reasonable alternatives is not an argument in favor of imposing no restriction at all and whenever a restriction is imposed that gives a lower bound to a firm's average marginal cost, one avoids the indeterminacy conclusion of Proposition 2.

Example 5. Consider the following observations of firms i and j:
(i) at observation t, $P_t = 14$, $Q_{i,t} = 50$ and $Q_{j,t} = 100$;
(ii) at observation t', $P_{t'} = 4$, $Q_{i,t'} = 60$ and $Q_{j,t'} = 120$.
We claim that these observations are not convincingly Cournot rationalizable.

Suppose instead that it is. By (2), we have

$$\bar{C}_1'(Q_{i,t}) \geq P_t \left[1 - \frac{Q_{i,t}}{Q_{j,t}}\right].$$

This gives $\bar{C}_i'(50) \geq 7$. The analogous inequality at t' gives $\bar{C}_i'(60) \geq 2$. By the convincing criterion, the added cost incurred by firm i as it increases output from 50 to 60 is at least $4.5 \times 10 = 45$. On the other hand, at observation t', the added revenue made by firm i as it increases output from 50 to 60 is no more than 40, which means that the firm is

[15] There is, of course, a strong case that 1/2 is the most natural choice since the cost function constructed using that coefficient approximates the Riemann integral of the marginal cost function and will in the limit tend towards the true cost function. A coefficient smaller or larger than 1/2 may lead (respectively) to an under- or over-estimate of the true cost function.

better off producing at 50 rather than 60 at t'. We conclude that these observations are not convincingly rationalizable.

Our final result sets out the necessary and sufficient conditions for a set of observations to be convincingly Cournot rationalizable. Note that these conditions take the form of a linear program so its implementation is not typically a complex issue.

THEOREM 6. *A set of observations* $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ *is convincingly Cournot rationalizable if, and only if, the following conditions are satisfied:*
(a) there are positive scalars $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$*, with* $\delta_{i,t} = \delta_{i,t'}$ *if* $Q_{i,t} = Q_{i,t'}$*, that has the common ratio property;*
(b) there are positive scalars $\{\Delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$*, with* $\Delta_{i,t} = \Delta_{i,t'}$ *if* $Q_{i,t} = Q_{i,t'}$*, that has the discrete marginal property, i.e.,*

$$\sum_{Q_{i,s}\in\mathbb{Q}(t',t)} \Delta_{i,s}\left(Q_{i,s} - Q_{i,l(s)}\right) < P_t(Q_t - Q_{t'}) \text{ for all } t' \in L_i(t)\text{; and} \tag{17}$$

(c) for all i *and* $t \neq t_i^*$*,*

$$\Delta_{i,t} \geq \frac{1}{2}\left[\delta_{i,l_i(t)} + \delta_{i,t}\right]. \tag{18}$$

Proof: To see that these conditions are necessary, set $\delta_{i,t} = \bar{C}_i'(Q_{i,t})$ and $\Delta_{i,t} = [\bar{C}_i(Q_{i,t}) - \bar{C}_i(Q_{i,l_i(t)})]/[Q_{i,t} - Q_{i,l_i(t)}]$. Then (c) is just the convincing criterion on the cost functions, (a) follows from the first order condition at the equilibrium and we have already explained why (b) (see the discussion following (13)).

For the sufficiency of the conditions, set

$$C_{i,t} = \sum_{Q_{i,s}\in\mathbb{Q}_i(t_i^*,t)} \Delta_{i,s}(Q_{i,s} - Q_{i,l(s)})\,;$$

then (17) guarantees that the data set $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ obeys the discrete marginal property (see (12) and (13)). Lemmas 5 and 6 then guarantee the existence of $\bar{P}_t$ and $\bar{C}_i$ with the desired properties. In particular, $\bar{C}_i(Q_{i,t}) = C_{i,t}$, $\bar{C}_i'(Q_{i,t}) = \delta_{i,t}$ and the average marginal cost between $Q_{i,l(t)}$ and $Q_{i,t}$, is $\Delta_{i,t}$, so that $\bar{C}_i$ satisfies the convincing criterion because of (18). QED

Corollary 1 says that there is a Cournot rationalization with C^1 convex cost curves if the common ratio property holds and $\delta_{i,t}$ is finely increasing with $Q_{i,t}$. Note that the latter condition guarantees the existence $\Delta_{i,t}$ obeying (b) and (c) in Theorem 6: set $\Delta_{i,t} = \delta_{i,t}$ and it is clear that (17) and (18) are satisfied. So Theorem 6 is certainly consistent with

Corollary 1. Of course, the interest of this result lies precisely in the fact that it potentially allows for cases where $\delta_{i,t}$ is *not* increasing with $Q_{i,t}$.

6. Allowing for Noisy Data

Revealed preference tests by their very nature are exact. Errors in the observed data could lead to a potential false rejection by the test even if the errors are small. Put differently, one might be concerned that our tests might reject the data due to small measurement errors even though underlying true data is consistent with Cournot behavior. An insight by Varian (1985) allows us to deal with such issues. In this section, we briefly describe how his approach can be adapted to our tests.

Once again, consider the case of single product where firms have convex cost functions. We denote the set of data sets which are Cournot rationalizable with convex cost functions by $\mathfrak{D}$. Formally,

$$\mathfrak{D} = \Big\{ \{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}} : \{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}} \text{ is Cournot rationalizable with convex cost functions} \Big\}.$$

Assume that we are given an observed data set $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ that is not Cournot rationalizable with convex cost functions, so $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}} \notin \mathfrak{D}$. Suppose that this data set has been contaminated with measurement error. In other words, the "true" data set is given by

$$\{[P_t + \varepsilon_t^P, (Q_{i,t} + \varepsilon_{i,t}^Q)_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}},$$

where the error terms $\Xi = \left\{ \varepsilon_t^P, (\varepsilon_{i,t}^Q)_{i\in\mathcal{I}} \right\}_{t\in\mathcal{T}}$ are assumed to be classical with variance σ^2. Consider now the null hypothesis that the true data set is Cournot rationalizable with convex cost functions. A test statistic for this null hypothesis can now be based on the loss function

$$\mathbb{L} = \frac{\boldsymbol{\Xi}}{\sigma^2},$$

where $\boldsymbol{\Xi}$ is the vector formed by concatenating the errors in the set Ξ. Since the errors are normally distributed the test statistic $\mathbb{L}$ will have a chi-squared distribution with degrees of freedom equal to the number of data points. Hence, we can find critical values for any desired level of significance and reject the null hypothesis if the test statistic is great than the critical value.

Since the errors are not observed, we can compute the test statistic by solving the following optimization problem:

$$\min_{\Xi} \left\{ \frac{\Xi}{\sigma^2} \right\},$$

$$\text{subject to } \{[P_t + \varepsilon_t^P, (Q_{i,t} + \varepsilon_{i,t}^Q)_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}} \in \mathfrak{D}.$$

This problem essentially involves finding the minimum perturbation to the observed data so that the perturbed data is Cournot rationalizable. The above constraint is simply the set of inequalities and equalities imposed in Theorem 1 and as a result it is a well defined numerical optimization problem.

It is worth pointing out that the variance of the measurement error is typically unknown. Varian (1985) suggests that estimates of the variance can be obtained from parametric or nonparametric fits of the data, from knowledge of how the variables were actually measured or from other data sources. Alternatively, he suggests computing how large the variance would need to be in order for the null hypothesis to be accepted and comparing this value to our prior opinions. If it is much smaller than our prior opinions regarding the precision with which the data has been measured, we may be compelled to accept the null.

Of course, the above procedure can be applied to all the generalizations of the basic result. We can consider the set $\mathfrak{D}$ to be the set of data sets which are Cournot rationalizable with cost shifters, $\theta - CV$ rationalizable, Cournot rationalizable with multiple products or convincing Cournot rationalizable.

7. Application: The world market for crude oil

Petroleum accounts for more than one third of global energy consumption, and in April 2009 world oil production was more than 72 million barrels per day (*Monthly Energy Review (MER)*, 2009). Accounting for roughly one third of global oil production, OPEC is a dominant player in the international oil market. OPEC was founded in 1960 and exists, in its own words, "to co-ordinate and unify petroleum policies among Member Countries, in order to secure fair and stable prices for petroleum producers; an efficient, economic and regular supply of petroleum to consuming nations; and a fair return on capital to those investing in the industry." OPEC rose to prominence during the energy crises of the 1970s for its embargo in response to Western support of Israel during the 1973 Yom Kippur War. Since the start of the 1980s, with the abolition of US price controls and increased production by the rest of the world, OPEC's influence on oil prices has declined. Beginning in 1982, OPEC began to allocate production quotas to its members, replacing a system of posted prices. This has not, however, permitted OPEC to dictate world prices, since the majority the world's oil is produced by non-members and the only sanction available to police its members is Saudi Arabia's spare capacity.

OPEC's stated aims are effectively those of a cartel, but its ability to set world oil prices is questionable. Hence, a large literature has emerged that attempts to model its actions and to test whether these models fit its observed behavior. For the most part, the literature suggests that OPEC is a "weakly functioning cartel" of some sort, and is not "competitive" in either the price-taking or non-cooperative Cournot senses – see, for example, Alhajji and Huettner (2000), Almoguera and Herrera (2007), Dahl and Yücel (1991), Griffin and Neilson (1994), or Smith (2005). Many of these tests rely on parametric assumptions about the functional forms taken by market demand, countries' objective functions and production costs. Typically, they also require that factors shifting the cost and inverse demand functions be observed, and rely on constructed proxies such as estimates of countries' extraction costs, the presence of US price controls, and involvement of an OPEC member in a war. Given the ambitious questions they are trying to answer, this seems unavoidable.

Our objective is more modest and more specific. All we wish to do is to use the results developed in the previous sections to test whether the behavior of the oil-producing countries is consistent with the Cournot model or, more generally (given the discussion in Section 3), any *symmetric* CV model. Our tests make use of very few ancillary assumptions, giving, so to speak, the greatest benefit of the doubt to the hypothesis, by allowing for a very large class of cost functions and by not making any assumptions at all about the evolution of demand (apart from the assumption that it is downward sloping with respect to output). In spite of this apparent permissiveness, our tests can reject the restrictions of the Cournot model in real world data.

Two sources of data are used for this study. The first is the *Monthly Energy Review (MER)*, published by the US Energy Information Administration. This provides full-precision series of monthly crude oil production in thousands of barrels per day by the twelve current OPEC members (Algeria, Angola, Ecuador, Iran, Iraq, Kuwait, Libya, Nigeria, Qatar, Saudi Arabia, the United Arab Emirates, and Venezuela) and seven non-members (Canada, China, Egypt, Mexico, Norway, the United States, and the United Kingdom).[16] This series also contains total world output. The data are available from

[16]Russia and the former Soviet Union are not used here, because the two are not comparable units. Although the composition of OPEC has changed over the course of the data (Ecuador left in 1994 and returned in 2007, Gabon left in 1995, Angola joined in 2007, and Indonesia left in 2007), the overall

January 1973 until April 2009, giving a total length of $T = 436$ months and $\bar{M} \times T = 8248$ country-month observations. There are only seven instances in the data in which an individual country's monthly production is zero,[17] and so false acceptances and rejections of the test due to violation of this assumption will be small in number. The second source of data is a series of oil prices published by the St. Louis Federal Reserve, in dollars per barrel. This series is deflated by the monthly consumer price index reported by the Bureau of Labor Statistics, so that prices are in 2009 US dollars. Since the time windows over which Cournot behavior is tested are short (twelve months or less), the adjustment for inflation should not matter to the results.[18]

Each test consists of using a linear programming algorithm to find whether there exists a solution to the specified linear program. If a solution exists, this subset of the data can be rationalized within the Cournot model, i.e. Cournot behavior by these M countries is supported by the data during the period tested. Clearly, as M and W increase, it is more likely that at least one country is not behaving optimally in at least one period, and so it is more likely that it will not be possible to satisfy the set of inequalities. Rather than performing a single test for whether the entire data series can be rationalized, we select a number of countries M, and then test whether the data for each of the $\binom{\bar{M}}{M}$ possible combinations of countries in each of the $T+1-W$ periods of length W can be rationalized. We then report the percentage of these $\binom{\bar{M}}{M} \times (T+1-W)$ cases in which optimal behavior is rejected. The time windows selected are short; W is either 3 months, 6 months, or 12 months. This is in keeping with the assumption that cost functions do not change over the period of the test. If a test is able to reject for a small amount of data (for example, three countries over three months), it demonstrates that, despite the generality of the non-parametric framework, the test has sufficient power to detect non-optimal behavior in real data.

We use the linear program specified in Theorem 1 to test whether the data sets are Cournot rationalizable with convex cost curves. Table 1 presents the percentage of cases (in the sense explained in the Notes below the table) for which the data is *not* Cournot rationalizable with convex costs over groups of 2, 3, 6 and 12 OPEC countries within windows of 3, 6, and 12 months. The results are unambiguous – once there are more than a handful of observations used for the test, the behavior of OPEC members cannot be

pattern of rejecting Cournot behavior below does not depend on what countries are considered to be part of OPEC. Reported tests consider groups of 2, 3, 6, and 12 member states; these overwhelmingly reject Cournot behavior for almost all country groups in periods longer than a few months.

[17] Ecuador, April, 1987; Iraq, February and March, 1991; Kuwait, February through April, 1991.

[18] The rejection rates reported below are similar with nominal price series.

Table 1: Rejection rates with convex cost functions

		OPEC sample			
		Number of Countries			
		2	3	6	12
	3 Months	0.28	0.54	0.89	1.00
Window	6 Months	0.65	0.89	1.00	1.00
	12 Months	0.90	0.99	1.00	1.00
		Non-OPEC sample			
		Number of Countries			
		2	3	6	7
	3 Months	0.44	0.75	0.99	1.00
Window	6 Months	0.83	0.98	1.00	1.00
	12 Months	0.96	1.00	1.00	1.00

Notes: The rejection rate reported is the proportion of cases that were rejected. For example, there are $436 + 1 - 3 = 434$ three month periods in the data. There are 66 possible combinations of two out of twelve OPEC members. The entry for two countries and three months, then, reports that out of the $434 \times 66 = 28,644$ possible tests of two OPEC members over three months, 8138, or 28% could not be rationalized.

explained by the Cournot model with convex costs. For nearly 90% of six-month periods with three countries, the test rejects optimal behavior. Once six countries are included, fewer than one six-month case in ten thousand can be rationalized. The same test was performed for the non-OPEC countries (see Table 1). Once again the results are strongly against the Cournot model. For almost all six month periods, when at least three countries are considered, the data cannot be rationalized by the Cournot model with convex costs.

It is, in principle, possible that the tests reported in Table 1 rejected the Cournot hypothesis because the convexity of the cost functions is too strong an assumption. To address this potential problem, tests for convincing Cournot rationalizability (which allows for non-convex costs) using the linear program specified in Theorem 6 were also carried out. These are reported in Table 2. Given the very permissive setup, one may expect the test to have little power, but that is not the case. Rejection rates for the countries in OPEC exceed 50% with 3 countries and 6 observations. In the case of the non-OPEC countries the drop in the rejection rate is sharper and the picture becomes mixed, with rejection exceeding 50% only with 6 countries and 6 observations.

Table 2: Rejection rates with convincing cost functions

OPEC sample

		Number of Countries			
		2	3	6	12
	3 Months	0.21	0.41	0.76	0.98
Window	6 Months	0.40	0.66	0.92	1.00
	12 Months	0.60	0.84	0.98	1.00

Non-OPEC sample

		Number of Countries			
		2	3	6	7
	3 Months	0.06	0.13	0.36	0.43
Window	6 Months	0.12	0.25	0.63	0.73
	12 Months	0.20	0.45	0.84	0.90

Notes: See Table 1.

Appendix

Proof of Lemma 6: For each firm i, define $g_i(q_i) = k_i(q_i - Q_{i,t}) + \delta_{i,t}$. The graph of g_i is a line, with slope k_i that passes through the point $(Q_{i,t}, \delta_{i,t})$. Since $\delta_{i,t} < P_t$, and $\bar{C}_i$ is C^2, there is $\epsilon > 0$ and k_i (for $i \in \mathcal{I}$) such that, $P_t > g_i(Q_{i,t} - \epsilon)$ and for q_i in the interval $[Q_{i,t} - \epsilon, Q_{i,t})$, we have

$$g_i(q_i) > \bar{C}_i'(q_i). \tag{19}$$

(Note that k_i must be a negative number if $\bar{C}_i''(Q_{i,t}) < 0$.) For q_i in $[0, Q_{i,t} - \epsilon]$, there exists $\zeta > 0$ such that

$$Pq_i - \bar{C}_i(q_i) < PQ_{i,t} - \bar{C}_i(Q_{i,t}) \text{ for } P_t < P < P_t + \zeta; \tag{20}$$

this follows from property (iii) in Lemma 5. Note that ζ is common across all firms.

We shall specify the function $\bar{P}_t'$, so $\bar{P}_t$ can be obtained by integration. Holding the output of firm j (for $j \neq i$) at $Q_{j,t}$, we denote the marginal revenue function for firm i by $\bar{m}_{i,t}$; i.e., $\bar{m}_{i,t}(q_i) = \bar{P}_t'(\sum_{j\neq i} Q_{j,t} + q_i)q_i + \bar{P}_t(\sum_{j\neq i} Q_{j,t} + q_i)$. We first consider the construction of $\bar{P}_t'$ in the interval $[0, Q_t]$, where $Q_t = \sum_{i\in\mathcal{I}} Q_{i,t}$. Choose $\bar{P}_t'$ with the following properties: (a) $\bar{P}_t'(Q_t) = (\delta_{i,t} - P_t)/Q_{i,t}$ (which is equivalent to the first order condition $\bar{m}_{i,t}(Q_{i,t}) = \bar{C}_i'(Q_{i,t}) = \delta_{i,t}$; note that there is no ambiguity here because of (1)), (b) $\bar{P}_t'$ is negative, decreasing and concave in $[0, Q_t]$, (c) $\int_0^{Q_t} \bar{P}_t'(q)dq = P_t - \bar{P}_t(0) > -\zeta$ and (d) $\bar{P}_t'(Q_t - \epsilon)$ is sufficiently close to zero so that $\bar{m}_{i,t}(Q_{i,t} - \epsilon) > g_i(Q_{i,t} - \epsilon)$. Property (b) guarantees that $\bar{m}_{i,t}$ is decreasing and concave (as a function of q_i). This fact, together with (a) and (d), ensures that $\bar{m}_{i,t}(q_i) > g_i(q_i)$ for all i and q_i in $[Q_{i,t} - \epsilon, Q_{i,t})$; combining with (19), we obtain $\bar{m}_{i,t}(q_i) > \bar{C}_i'(q)$. Therefore, in the interval $[Q_{i,t} - \epsilon, Q_{i,t}]$, firm i's profit is maximized at $q_i = Q_{i,t}$. Because of (c), $P_t < \bar{P}_t(q) < P_t + \zeta$, so by (20), $\bar{P}_t(\sum_{j \neq i} Q_{j,t} + q_i)q_i - \bar{C}_i(q_i) < P_t Q_{i,t} - \bar{C}_i(Q_{i,t})$ for q_i in $[0, Q_{i,t} - \epsilon]$.

To recap, we have constructed $\bar{P}_t'$ (and hence $\bar{P}_t$) such that, with this inverse demand function, firm i's profit at $Q_{i,t}$ is higher than at any output below $Q_{i,t}$, so long as other firms are producing $\sum_{j \neq i} Q_{j,t}$. Our next step is to show how to specify $\bar{P}_t'$ for $q > Q_t$ in such a way that firm i's profit at $q_i = Q_{i,t}$ is higher than at any output level above $Q_{i,t}$ (for every firm i). It suffices to have $\bar{P}_t$ such that, for $q_i > Q_{i,t}$,

$$\bar{m}_{i,t}(q_i) = \bar{P}_t'(\sum_{j \neq i} Q_{j,t} + q_i)q_i + \bar{P}_t(\sum_{j \neq i} Q_{j,t} + q_i) < \bar{C}_i'(q_i),$$

so firm i's marginal cost always exceeds its marginal revenue for $q_i > Q_{i,t}$. Provided $\bar{P}_t$ is decreasing, it suffices to have $\bar{P}_t'(\sum_{j \neq i} Q_{j,t} + q_i)q_i + P_t < \bar{C}_i'(q_i)$, which is equivalent to

$$-\bar{P}_t'(\sum_{j \neq i} Q_{j,t} + q_i) > \frac{P_t - \bar{C}_i'(q_i)}{q_i} \text{ for all firms } i.$$

This can be re-written as

$$-\bar{P}_t'(Q_t + x) > \frac{P_t - \bar{C}_i'(Q_{i,t} + x)}{Q_{i,t} + x} \text{ for } x > 0 \text{ and all firms } i \tag{21}$$

The right side of this inequality is a finite collection of continuous functions of x and at $x = 0$, the two sides are equal to each other (because of (1)). Clearly we can choose $\bar{P}_t' < 0$ such that (21) holds for $x > 0$. QED

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