On revealed preferences in oligopoly games

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Abstract

We consider the following theoretical problem: a finite number of observations of a homogeneous-good oligopoly market are made, and in each observation we observe firms' prices, outputs and possibly cost information- what restrictions must the observations satisfy for each observation to be rationalized as a pure strategy Bertrand equilibrium? We provide a complete characterization of price/output/cost observations which can be Bertrand rationalized and some partial characterizations of price/output observations. The conditions which characterize the sets are economically intuitive and take the form of linear inequalities. Moreover, together with recent results established by Carvajal and Quah (2009), we can characterize which homogeneous-good market observations are consistent with either the Bertrand or Cournot oligopoly models.

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1 Introduction

Suppose we make a finite set of observations $T = \{1, ..., m\}$, $m \geq 1$, of a perfectly homogeneous-good oligopoly market. There is a finite number of firms $N = \{1, ..., n\}$, $n \geq 2$, which compete in the market. In each observation we observe each firm's price, their output, and possibly their cost information. Given this information how should we go about checking whether these observations are consistent/inconsistent with firms playing a Nash equilibrium in prices? That is to say, if each firm simultaneously and independently chooses a price, with a commitment to supply all the demand forthcoming at that price, a classical Bertrand game, what observable restrictions does the Nash equilibrium in prices impose upon the outcomes? In this paper we solve this theoretical problem by providing a complete characterization of the observations which can be 'rationalized' by the classical Bertrand model. We identify two main conditions which are economically meaningful and take the form of linear inequalities. The first of these conditions is what we term the *monopoly deviation condition* which requires that we do not observe a situation where one firm could profitably deviate by serving the entire market demand. The second condition is what we term the *tie deviation condition* which requires that we do not observe a situation where one firm could profitably deviate by joining a price tie. Together with a weak monotonicity condition on observed costs we find that these conditions provide a complete characterization of the observations which can be Bertrand rationalized with standard primitives.

This paper continues the tradition in economic theory of analyzing what structure, if any, various equilibrium concepts impose upon observable outcomes. Although the preference based approach to economic theory, where we state specific primitives and analyze equilibrium existence and comparative statics etc., is still the way economic theory is mainly done, the establishment of observable restrictions provides a useful complement to primitive-based theory for a number of reasons. First, it allows us to establish whether economic equilibrium concepts impose any structure upon the observable outcomes and therefore whether it is possible, even at a theoretical level, to refute equilibrium concepts. Second, by asking whether a set of observations can be revealed consistent/inconsistent with a given equilibrium concept we often permit greater variety of economic primitives than when we directly state primitives and analyze equilibrium properties. For example, when analyzing classical Bertrand games it is typical to assume that cost functions of firms are convex. Here we impose no such restrictions. Instead, we simply require that observations are consistent with an increasing cost function. Finally, the structure of equilibrium sets is a useful addition to the stock of theoretical knowledge regarding canonical economic models.

There is a substantial literature on revealed preferences in economic models so we shall only mention the key touchstones in the literature here. The most frequent recourse to revealed preference is in the context of consumer theory where Samuelson's weak axiom of revealed preference is well-known to constitute a necessary condition for the existence of a 'rationalizing' preference relation. That is to say, for those observations which violate the weak axiom it is not possible to find a utility function which would generate the observations from utility maximization. An early counter example, by John R. Hicks, showed that in a three-commodity economy consumer's Walrasian demands may satisfy the weak axiom and still exhibit intransitive choices (and therefore be revealed inconsistent with utility maximization).[1] Afriat (1967) was a seminal paper in that it looked for necessary and sufficient conditions upon price/quantity observations for the existence of a rationalizing utility function. The key condition was a strengthening of the weak axiom, a form of cyclical consistency, which is equivalent to Hendrik Houthakker's strong axiom of revealed preference.[2]

This early literature on revealed preferences in consumption theory clearly illustrates the method of characterizing observable restrictions. The method starts from the basis that there is something unobservable (preference) which is fixed across observations, and something which is observable (budget sets/choices) which change across observations. The literature then asks what restrictions must be placed on the observables for them to be consistent which the existence of a well-behaved unobservable. Following in this tradition we shall say that a set of oligopoly observations is Bertrand rationalizable if there exists a market demand for each observation and a cost function for each firm which is able to account for the observed market outcomes. Hence, the analogue with consumer theory is that the cost function represents the type of each of the firms which is fixed across observations whereas the changes in the market demand represent the observable which accounts for the changes in observed outcomes.

The canonical general equilibrium model which brings consumption choices together with market clearing conditions was thought to have few observable restrictions because of the well-known result that generically economies have a finite, and cardinally odd, set of equilibria but no other general restrictions are imposed upon the equilibrium price set. The results of Debreu-Mantel-Sonnenschein went further by showing that for any bounded continuous function defined on a compact subset of the price space (not including the origin) which satisfies Walras law and is homogeneous of degree zero in prices there exists an economy

[1]See, for example, Mas-Colell et al. (1995, p.35). Although observations which satisfy the weak axiom of revealed preference also satisfy the weak weak axiom of revealed preference and can be rationalized by a complete, but not necessarily transitive, preference relation (Jerison and Quah, 2008).

[2]Forges and Minelli (2009) have extended Afriat's theorem to budget sets which may have nonlinear frontiers such as in strategic market games.

which generates the function as the excess demand.[3] For a long time these results were thought to seriously undermine the usefulness of the general equilibrium model. However, in an important paper Brown and Matzkin (1996) analyzed what restrictions observations of prices and individual endowments must satisfy to belong to the equilibrium manifold.[4] That is to say, what restrictions must prices and endowments satisfy so that there exist quasiconcave utility functions for each trader such that the observed prices and endowments are points on the equilibrium manifold generated by those preferences. They showed that there exist non-trivial restrictions and exhibited simple exchange economies which could not be rationalized as Walrasian equilibria.[5]

There is however a fundamental difference between this paper and the seminal works of Afriat (1967) and Brown and Matzkin (1996) in that the latter were characterizing the observable outcomes of non-strategic equilibrium concepts whereas this paper is concerned with the strategic Nash equilibrium. It is only in the last ten years or so that research has turned to applying the revealed preference method to the Nash equilibrium.[6] However, as the Nash equilibrium is a full rationality concept it is perhaps not surprising that it should be amenable to characterization in terms of observables. This more recent literature is what is most closely related to this paper. Sprumont (2000) considered the case where we observe players choices from all possible subsets of strategy choices, all action spaces were finite, and identified two conditions, expansion and contraction consistency, which are necessary and sufficient to be able to rationalize the observed choices as Nash equilibria.[7] Zhou (2005) considered two-player games where players' strategy sets were the unit interval and assumed that we observe a finite subset of choices. She then asked what conditions the observed choices must satisfy to be able to find payoff functions for the players which are quasiconcave in own strategy and continuous in all strategies which would explain the observed choices as Nash equilibria. By exploiting the path connectedness of the best response correspondence she provided a no-improper-crossing condition which if satisfied means the observations admit rationalizing payoff functions. Most closely related to this paper is work by Carvajal and

[3]This remains true even if one restricts attention to distribution economies where individual endowments are collinear and the distribution of income is independent of prices (Kirman and Koch, 1986).

[4]The equilibrium manifold being the graph of the Walras correspondence.

[5]Although as Rizvi (2006, p.239) notes the Brown and Matzkin results do not overturn Kenneth Arrow's statement that "in the aggregate, the hypothesis of rational behaviour has in general no implications" because their results rely on us being able to observe *individual* endowments.

[6]A survey of the recent developments of testable restrictions in both equilibrium theory and game theory is provided by Carvajal et al. (2004).

[7]Anticipating this work, Sprumont (2000, p.221) noted at the end of his paper that it would be interesting to characterize observable restrictions in games with more "economic flesh" such as oligopoly games.

Quah (2009) which considered the Cournot oligopoly model. They assumed we make a finite number of observations of a homogeneous-good oligopoly market and in each observation we observe a single market price, firms' outputs and possibly also cost information. They show that if the observations satisfy a marginal condition, which roughly means that we do not observe instances where firms can profitably deviate by reducing their outputs, then providing observed costs are co-monotone with outputs, the observations can be rationalized as Cournot equilibria. If cost information is unavailable, then any set of observations can be Cournot rationalized if one permits a general increasing cost function. However, they introduce a stronger criterion of a 'convincing rationalization', which means that the cost function must be constructed so that the marginal cost lies between the observed marginal costs, and show that this imposes restrictions upon price/output observations.

This paper aims to add to this literature by examining the other benchmark oligopoly model. We start from the same point as Carvajal and Quah (2009) by assuming that we observe firms' prices, outputs and possibly cost information and look for the restrictions which the Bertrand equilibrium imposes upon the observables. The case when firms have the same price in a given observation could clearly also be consistent with firms choosing outputs in equilibrium. This special case is one of the most interesting aspects of the results provided here as we shall be able to perform revealed preference analysis on example observations to establish their consistency with either the Bertrand or Cournot equilibrium concepts. However, given the idealized nature of the Bertrand model it is unlikely that the conditions provided here could be used to test real-world market competition. Nevertheless, given the prominence of the model in the literature we consider the characterizations to be of considerable theoretical interest. In the next section of the paper we set out the general theoretical problem and provide the major results. The mathematical requirements are limited as we only use basic set theory to organize the observations. In the next section we present a couple of simple examples and use them to apply the theoretical results. The final section draws some conclusion regarding future directions for research in this area.

2 Revealed Nash equilibria in oligopoly games

Before addressing the main problem of observable restrictions which the Nash equilibrium imposes upon oligopoly outcomes we shall first define the standard notions of Bertrand and Cournot equilibrium as well as when a set of observations admit a rationalization by either of the models. Although there is a vast literature regarding the two models, the aim here is to present the simplest form of the equilibrium concept. First consider a perfectly homogeneous-good market with $N = \{1, ..., n\}$, $n \geq 2$, firms. Each firm has a cost function $C_i : \Re_+ \to \Re_+$ which is strictly increasing, continuous and satisfies $C_i(0) = 0$. The market demand $D : \Re_+ \to \Re_+$ is continuous and strictly decreasing whenever $D > 0$. We shall let the function $F : \Re_+ \to \Re_+$ denote the inverse market demand in what follows. Now consider first the notion of a Bertrand equilibrium. Those wanting a more detailed exposition of the model are referred to Baye and Kovenock (2008) or Vives (1999, Ch.5). Each firm simultaneously and independently chooses a $P_i \in \Re_+$. Firms in the market commit to supplying all the demand forthcoming at any price.[8] If a firm posts the unique minimum price in the market then it obtains all the market demand and its payoff is $P_i D(P_i) - C_i(D(P_i))$. If a firm ties with $m - 1$ firms at the minimum price then they share the demand equally between themselves and the payoff of each firm is given by $\frac{1}{m} P_i D(P_i) - C_i(\frac{1}{m} D(P_i))$. If a firm is undercut by any other firm in the market then it obtains zero demand, and given the assumption that the cost function passes through the origin, its payoff is zero. These payoffs are summarized below.

$$\pi_i(P_i, P_{-i}) = \begin{cases} P_i D(P_i) - C_i(D(P_i)) & \text{if } P_i < P_j \ \forall j \neq i; \\ \frac{1}{m} P_i D(P_i) - C_i(\frac{1}{m} D(P_i)) & i \text{ ties with } m-1 \text{ firms;} \\ 0 & \text{if } P_i > P_j \text{ for some } j. \end{cases} \tag{1}$$

Given this specification of the payoffs we can state the Bertrand equilibrium as a Nash equilibrium of this price-setting game.

Definition 1 *A pure strategy Bertrand equilibrium is a Nash equilibrium of the game with payoffs defined by eq.(1). That is, a vector of prices* (P_i^B, P_{-i}^B) *such that* $\pi_i(P_i^B, P_{-i}^B) \geq \pi_i(P_i, P_{-i}^B)$ *for all* $P_i \in \Re_+$ *and* $i \in N$.

Now consider the alternative case when firms choose outputs which are sent to the market. For a more detailed exposition of the Cournot model the reader is referred to Vives (1999. Ch.4). Each firm simultaneously and independently chooses a $Q_i \in \Re_+$. Given the total output chosen by the firms the market demand clears this output and sends back a single market-clearing price. The payoff which any firm receives, given the vector of chosen outputs is (Q_i, Q_{-i}), is given below.

$$\pi_i(Q_i, Q_{-i}) = F(\sum_{j=1}^{n} Q_j) Q_i - C_i(Q_i) \tag{2}$$

The Cournot equilibrium is then a Nash equilibrium of this output-setting game.

[8] This is what distinguishes Bertrand competition from Bertrand-Edgeworth competition.

Definition 2 *A pure strategy Cournot equilibrium is a Nash equilibrium of the game with payoffs defined by eq.(2). That is, a vector of outputs* (Q_i^C, Q_{-i}^C) *such that* $\pi_i(Q_i^C, Q_{-i}^C) \geq \pi_i(Q_i, Q_{-i}^C)$ *for all* $Q_i \in \Re_+$ *and* $i \in N$.

These are the the most commonly used equilibrium concepts in oligopoly theory and yet little has been known, until recently, about what structure they impose upon the observable outcomes in the market. We now turn to this problem. Suppose we make a finite number of observations, $T = \{1, ..., m\}$, $m \geq 1$, of a homogenous-good oligopoly market. In each period we observe the price of each firm, P_{it}, their output, Q_{it}, and their costs incurred, C_{it}. The total set of observations can then be summarized as $(P_{it}, Q_{it}, C_{it})_{i\in N, t\in T}$. To organize the observations let $P_t^* = \min_{i\in N} P_{it}$. That is, P_t^* is the minimum price posted in the market in observation t. Let $Q_t^* = \sum_{i\in N} Q_{it}$ denote the aggregate output produced in observation t. The set of firms which tie at the minimum price, what we shall informally call the set of 'active firms', is given by $A_t = \{i \in N : P_{it} = P_t^*\}$.

We shall impose certain restrictions upon the type of observations which we make to ensure that they are not immediately inconsistent with the models. For example, if we were to observe an observation where firms have different prices then such a set of observations is clearly never going to be consistent with firms choosing outputs and the market sending back a single market-clearing price. Similarly, if we make observations where two firms produce output at different prices, so the law of one price fails, then such a set of observations is always going to be inconsistent with the Bertrand model which postulates that all trade takes place at the minimum price. Therefore we shall introduce the notion of a *generic homogeneous-good set of oligopoly observations* which has the basic features of a perfectly homogeneous-good market.

Definition 3 *A set of oligopoly observations is a generic homogeneous-good market data set if it satisfies the following conditions:*
i) $P_{it} > 0$, $Q_{it} \geq 0$, $C_{it} \geq 0$, $P_{it}Q_{it} \geq C_{it}$ *and* $Q_{it} \neq Q_{it'}$ *whenever* $t \neq t'$.
ii) If $P_{it} > P_{jt}$ *then* $Q_{it} = 0$.
iii) If $P_{it} = P_{jt} = P_t^*$ *then* $Q_{it} = Q_{jt}$.
iv) $|A_t| \geq 2$.

The first part imposes mild conditions that we observe positive prices, non-negative outputs, revenues are greater than costs and firms' outputs change across observations. The requirement that outputs change across observations means we observe some variation in the data and it stops the data contradicting itself. The second part means that the market is consistent with the law of one price: if any firms raises their price above that of another firm then they receive zero demand and produce zero output. The third part states that firms tieing at the minimum price split the demand equally. In the literature on price games, alternative sharing rules at minimum price ties have been used, but as this is the most common rule used in the literature we shall consider this case here. The final part states that we do not observe a monopolist in any observation. A special type of data set is what we shall term a *single-price data set*: this is a generic homogeneous-good data set with the additional property that $P_{it} = P_t^*$ for all $i \in N$ and $t \in T$. This special case where we observe a single price by all firms in each observation is particularly interesting from a theoretical perspective because we shall be able to analyze whether observations are consistent/inconsistent with either the Bertrand or Cournot equilibrium concepts.

When a set of observations is consistent with firms playing a Nash equilibrium in prices or outputs we shall term the observations as rationalizable by the Bertrand or Cournot models respectively. As noted at the beginning, we shall assume that variations in observed outcomes are due to changes in the market demand across observations with firms' cost functions fixed across observations. The method of revealed preference starts from the basis that there is something observable which changes across observations and something unobservable which remains fixed across observations. The method then asks what restrictions must be placed upon the observables such that they are consistent with the existence of a well-behaved unobservable. Formally, we define the notion of Bertrand rationalizability below.

Definition 4 *A set of generic homogeneous-good observations, $(P_{it}, Q_{it}, C_{it})_{i \in N, t \in T}$, is Bertrand rationalizable if there exist C^2 functions, $\bar{C}_i : \Re_+ \to \Re_+$ for each $i \in N$, $\bar{D}_t : \Re_+ \to \Re_+$ for each $t \in T$, such that:*
i) $\bar{C}_i(0) = 0$ and $\bar{C}_i'(x) > 0$ for all $x > 0$.
ii) $\bar{D}_t(x) \geq 0$ and $\bar{D}_t'(x) \leq 0$ with the latter inequality holding strictly whenever $\bar{D}_t(x) > 0$.
iii) $\bar{C}_i(Q_{it}) = C_{it}$ and $\bar{D}_t(P_t^*) = Q_t^*$.
iv) The set of observed prices $(P_{1t}, ..., P_{nt})$ is a Bertrand equilibrium in pure strategies for each $t \in T$.

The first three parts state that we can find standard demands for each observation and cost functions for each firm such that the cost and demand functions explain the observed demands and costs. The final part states that these cost and demand functions must also be such that the observed set of prices in each observation constitute a pure strategy Bertrand equilibrium. The alternative concept of Cournot rationalizability requires that we can find inverse demands for each observation and cost functions for each firm such that these functions explain the observed outputs and costs. Moreover, the cost and inverse demand functions must be such that the observed outputs in each observation constitute a pure strategy

Cournot equilibrium. For completeness we state this formally below.

Definition 5 *A single-price data set, $(P_t^*, Q_{it}, C_{it})_{i\in N, t\in T}$, is Cournot rationalizable if there exist C^2 functions, $\bar{C}_i : \Re_+ \to \Re_+$ for each $i \in N$, $\bar{F}_t : \Re_+ \to \Re_+$ for each $t \in T$, such that:*
i) $\bar{C}_i(0) = 0$ and $\bar{C}_i'(x) > 0$ for all $x > 0$.
ii) $\bar{F}_t(x) \geq 0$ and $\bar{F}_t'(x) \leq 0$ with the latter inequality holding strictly whenever $\bar{F}_t(x) > 0$.
iii) $\bar{C}_i(Q_{it}) = C_{it}$ and $\bar{F}_t(Q_t^*) = P_t^*$.
iv) The set of observed outputs $(Q_{1t}, ..., Q_{nt})$ is a Cournot equilibrium in pure strategies for each $t \in T$.

Before stating conditions which characterize the rationalizable sets we shall introduce some notation which helps organize the observations. Let $R_i(t) = \{t' \in T : Q_{it'} \geq Q_t^*\}$. That is, $R_i(t)$ is the set of observations when the output of firm i is greater than the aggregate output in observation t. Let $S_i(t) = \{t' \in T : Q_{it'} < Q_{it}\}$. The set $S_i(t)$ is those observations when the output of firm i is less than its own output in observation t. We shall also want to compare firms' outputs with regards to the following quantity $\hat{Q}_t = Q_t^*/(|A_t| + 1)$. We introduce $M_i(t) = \{t' \in T : Q_{it'} \geq \hat{Q}_t\}$ which is the set of observations when the output of firm i is greater than or equal to $\hat{Q}_t$.

Definition 6 *A set of generic homogeneous-good oligopoly observations, $(P_{it}, Q_{it}, C_{it})_{i\in N, t\in T}$, satisfy the increasing cost condition (ICC) if, whenever $S_i(t) \neq \emptyset$, then $C_{it} - C_{it'} > 0$ for all $t' \in S_i(t)$.*

The interpretation of the increasing cost condition is straightforward: it states that whenever we observe a firm producing a higher output then its observed costs should increase. It should be clear that this is a necessary condition for rationalization by either oligopoly model. If violated then we would not be able to construct an increasing cost function which explains the observed costs.

Definition 7 *A set of generic homogeneous-good oligopoly observations, $(P_{it}, Q_{it}, C_{it})_{i\in N, t\in T}$, satisfy the monopoly deviation condition (MDC) if, whenever $R_i(t) \neq \emptyset$, then $P_t^* Q_{it} - C_{it} \geq P_t^* Q_t^* - C_{it'}$ for all $t' \in R_i(t)$ with the inequality holding strictly whenever $Q_{it'} > Q_t^*$.*

The monopoly deviation condition requires that the observed profit of any firms is at least as great as the profit they could obtain from supplying the entire market demand and incurring a cost at least as large as that required to meet the demand. Note that if the relevant inequalities are not defined then this does not violate the condition. The key point is that we do not observe a violation of the condition. The intuition behind the condition is that when it is satisfied we shall be able to find cost and demand function which are consistent with firms not wanting to post a price which undercuts the observed minimum price.

Definition 8 *A set of generic homogeneous-good oligopoly observations,* $(P_{it}, Q_{it}, C_{it})_{i \in N, t \in T}$*, satisfy the tie deviation condition (TDC) if, whenever* $M_i(t) \neq \emptyset$*, then* $P_t^* \hat{Q}_t \leq C_{it'}$ *for all* $t' \in M_i(t)$ *and* $i \in N \backslash A_t$ *with the inequality holding strictly whenever* $Q_{it'} > \hat{Q}_t$*.*

The tie deviation condition states that if a firm is not active in a given observation then the revenue it could obtain from tieing at the minimum price, note that $\hat{Q}_t$ is the share of the demand which the firm could obtain from tieing at the minimum price, is less than the cost which the firm would obtain from supplying a quantity at least as high as $\hat{Q}_t$. As with the monopoly deviation condition, if the relevant inequalities are not defined then this does not violate the condition. All that matters is that we do not observe a violation when the relevant quantities are defined. The intuition behind the condition is that if satisfied we shall be able to construct cost and demand functions such that inactive firms cannot profitably deviate by joining a minimum price tie.

Definition 9 *A single-price data set,* $(P_t^*, Q_{it}, C_{it})_{i \in N, t \in T}$*, satisfies the marginal condition (MC) if, whenever* $S_i(t) \neq \emptyset$*, then* $P_t^* Q_{it'} - C_{it'} < P_t^* Q_{it} - C_{it}$ *for all* $t' \in S_i(t)$*.*

The marginal condition applies to single-price data sets and says we do not observe a period where one firm could reduce its output and increase its profits provided that the market price remains unchanged. As with the previous conditions, if the inequality is not defined this does not contradict the condition. We now present the first main result which is a complete characterization of which generic homogeneous-good data sets can be Bertrand rationalized.

Theorem 1 *A set of generic homogeneous-good oligopoly observations,* $(P_{it}, Q_{it}, C_{it})_{i \in N, t \in T}$*, is Bertrand rationalizable if and only if it satisfies ICC, MDC and TDC.*

Proof. First we prove the necessity part of the result and second the sufficiency part. It should be clear from the definition of Bertrand rationalizability that if ICC is violated we cannot construct a cost function which satisfies part (i) and (iii) of Definition 4. Suppose instead that MDC is violated. Then there is an i, t and $t' \in R_i(t)$ such that $P_t^* Q_{it} - C_{it} \leq P_t^* Q_t^* - C_{it'}$ and $Q_{it'} > Q_t^*$. If the observations are rationalizable we have that $P_t^* Q_{it} - \bar{C}_i(Q_{it}) \leq P_t^* \bar{D}_t(P_t^*) - \bar{C}_i(Q_{it'})$. Then firm i could set a price $P_t^* - \epsilon$, and by choosing $\epsilon > 0$ to be sufficiently small, we have $\bar{D}_t(P_t^* - \epsilon) < Q_{it'}$ and $\bar{C}_i(\bar{D}_t(P_t^* - \epsilon)) < \bar{C}_i(Q_{it'})$. This means firm i could obtain a strictly higher profit by deviating to price $P_t^* - \epsilon$ and contradicts the observed prices being a Bertrand equilibrium.[9] Suppose that TDC is violated. Then there is an i, t and $t' \in M_i(t)$ such that $i \in N \setminus A_t$, $P_t^* \hat{Q}_t - C_{it'} \geq 0$ and $Q_{it'} > \hat{Q}_t$. If the observations are rationalizable we have $P_t^* \hat{Q}_t - \bar{C}_i(Q_{it'}) \geq 0$. As $i \in N \setminus A_t$ the firm obtains zero observed

[9]The same argument applies when $P_t^* Q_{it} - C_{it} < P_t^* Q_t^* - C_{it'}$ and $Q_{it'} = Q_t^*$.

profit in period t, but this means firm i could deviate and join the minimum price tie at P_t^* and obtain $P_t^* \hat{Q}_t - \bar{C}_i(\hat{Q}_t) > 0$ as $\bar{C}_i(\hat{Q}_t) < \bar{C}_i(Q_{t'})$.[10]

To see that the conditions in the theorem are also sufficient we first show how to construct firms' cost functions given the observed costs. Second we show how to construct the market demands. Finally we show that the constructed functions satisfy all the requirements of Definition 4. In constructing the relevant functions it is helpful to introduce the following notation. We shall let $r_i(t) = \{t' \in R_i(t) : Q_{it'} < Q_{it} \, \forall t \in R_i(t)\}$ so that $r_i(t)$ is that observation in $R_i(t)$ which minimizes the output of firm i across the observations in $R_i(t)$. We shall let $s_i(t) = \{t' \in S_i(t) : Q_{it'} > Q_{it} \, \forall t \in S_i(t)\}$. That is, $s_i(t)$ is that observation belonging to $S_i(t)$ which maximizes the output of firm i across the observations in $S_i(t)$. Let $m_i(t) = \{t' \in M_i(t) : Q_{it'} < Q_{it} \, \forall t \in M_i(t)\}$. That is, $m_i(t)$ is that observation belonging to $M_i(t)$ which minimizes the output of firm i across the observations in $M_i(t)$.

First, as the observed costs satisfy ICC we can construct a smooth, strictly increasing cost function for each firm with the properties that $\bar{C}_i(Q_{it}) = C_{it}$ and $\bar{C}_i(0) = 0$. Now we impose the following three restrictions upon these cost functions:

(1) For all $t \in T$ such that $R_i(t)$ is non-empty and $Q_{ir_i(t)} > Q_t^*$ choose the cost function so that $\bar{C}_i(Q_t^*) > \max\{P_t^*(Q_t^* - Q_{it}) + C_{it}, C_{is_i(t)}\}$.

(2) Define $V_i = \{t' \in T : R_i(t') = \emptyset\}$, $v_i = \{t' \in T : Q_{t'}^* \leq Q_t^* \forall t \in V_i\}$ and $w_i = \max_{t \in V_i} P_t^*(Q_t^* - Q_{it}) + C_{it}$. Then the cost functions are constructed so that $\bar{C}_i(Q_{v_i}^*) > w_i$.

(3) For all $i \in N \setminus A_t$ and $Q_{im_i(t)} > \hat{Q}_t$ construct the cost function so that $\bar{C}_i(\hat{Q}_t) > P_t^* \hat{Q}_t$.

The restriction in (1) can always be satisfied because as MDC is satisfied we have that $P_t^* Q_{it} - C_{it} > P_t^* Q_t^* - C_{ir_i(t)}$ and the cost function can be chosen so that $\bar{C}_i(Q_t^*) = C_{ir_i(t)} - \epsilon$ provided $\epsilon > 0$ is sufficiently small. The restriction in (2) can always be satisfied because it states that for those observations when we do not observe the firm i producing an output at least as large as the aggregate market output we can choose the cost function so that MDC is satisfied. The restriction in (3) can be satisfied because as TDC is not violated we have that whenever $i \in N \setminus A_t$ then $P_t^* \hat{Q}_t - C_{im_i(t)} < 0$ whenever $Q_{im_i(t)} > \hat{Q}_t$ and the cost function of firm i can be constructed so that $\bar{C}_i(\hat{Q}_t) = C_{im_i(t)} - \epsilon$ provided $\epsilon > 0$ is sufficiently small.

To see how to construct the market demands for each observation consider the following piecewise-affine market demand $\bar{D}_t(P_t) = \max\{0, 2Q_t^* - Q_t^* P_t / P_t^*\}$. This is strictly decreasing on the interior of the output space and satisfies $\bar{D}(P_t^*) = Q_t^*$.[11] Note that the revenue as a function of price is $R(P_t) = 2Q_t^* P_t - Q_t^* P_t^2 / P_t^*$. The marginal revenue is $R'(P_t) = 2Q_t^* - 2Q_t^* P_t / P_t^*$. Therefore $R'(P_t^*) = 0$ and revenue is maximized at P_t^*. This means

[10] The same argument applies when $P_t^* \hat{Q}_t - C_{it'} > 0$ and $Q_{it'} = \hat{Q}_t$.

[11] The kink which occurs when demand becomes zero could be smoothed out on a small interval to give a C^2 function.

$P_t\bar{D}_t(P_t) < P_t^*\bar{D}_t(P_t^*)$ whenever $P_t \neq P_t^*$. This property will be useful in showing that these functions are sufficient for Bertrand rationalizability.

To see that these functions are sufficient for Bertrand rationalizability note that we have satisfied parts $(i)-(iii)$ of Definition 4 and that all that remains to be shown is that given the constructed functions the set of observed market prices constitutes a pure strategy Bertrand equilibrium. Consider any firm $i \in A_t$ or $i \in N \setminus A_t$ increasing their price. As $|A_t| \geq 2$ they lose any demand and make zero profit which cannot be an improvement upon the observed profit. Second, consider a firm $i \in N \setminus A_t$ which lowers their price to tie at P_t^*. As the constructed cost function satisfies $P_t^*Q_t - \bar{C}_i(\hat{Q}_t) \leq 0$ this cannot be a improvement upon zero profit. Finally, suppose a firm undercuts the market and posts a price $P_t^* - \epsilon$, $\epsilon > 0$. The profit the firm would obtain is $(P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) - \bar{C}_i(\bar{D}_t(P_t^* - \epsilon))$. As the market demand achieves maximum revenue at P_t^* we have $(P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) < P_t^*\bar{D}_t(P_t^*) = P_t^*Q_t^*$. As the market demand is strictly decreasing and cost functions strictly increasing we have $\bar{C}_i(\bar{D}_t(P_t^* - \epsilon)) > \bar{C}_i(Q_t^*)$. Combining these two inequalities gives $(P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) - \bar{C}_i(\bar{D}_t(P_t^* - \epsilon)) < P_t^*Q_t^* - \bar{C}_i(Q_t^*)$. As the constructed cost functions satisfy $P_t^*Q_{it} - \bar{C}_i(Q_{it}) \geq P_t^*Q_t^* - \bar{C}_i(Q_t^*)$ we have $P_t^*Q_{it} - \bar{C}_i(Q_{it}) > (P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) - \bar{C}_i(\bar{D}_t(P_t^* - \epsilon))$. We can conclude that no firm can profitably undercut the market and the set of observed prices constitutes a pure strategy Bertrand equilibrium. ■

When we have the special case of a single-price data set this could be consistent with firms choosing outputs and letting the market determine a single-market clearing price. It turns out that the structure imposed by the Cournot equilibrium upon the observable market outcomes can be characterized by the increasing cost condition and the marginal condition.

Theorem 2 *A single-price data set, $(P_t^*, Q_{it}, C_{it})_{i \in N, t \in T}$, is Cournot rationalizable if and only if it satisfies ICC and MC.*

Proof. See Carvjal and Quah (2009, p.14, Thereom 1).[12] ■

The result in Theorem 2 actually applies to a broader class of data sets than that considered here in that one could allow firms to produce different quantities and the result still holds. It is worth noting that the characterizations in Theorem 1 and 2 are quite different in terms of what they impose upon the observables. The key condition for a set of observations to be Bertrand rationalizable is that we do not observe a situation where a firm can benefit from increasing its output and serving the entire market demand (monopoly deviation) whereas the condition for Cournot rationalizability is that we do not observe a situation where a firm can benefit from decreasing its output (marginal condition). Despite it often

[12] They assume that the data satisfies the increasing cost condition and therefore state the result in solely in terms of the marginal condition.

being assumed that price competition will result in lower prices than quantity competition it is well-known that when the set of Bertrand equilibria of a market is non-empty then it is possible for the Cournot equilibrium price to also constitute a Bertrand equilibrium (Vives, 1999, p.122). Therefore it is interesting to establish which market outcomes are consistent with both of the oligopoly models. We provide the following characterization.

Theorem 3 *A single-price data set, $(P_t^*, Q_{it}, C_{it})_{i \in N, t \in T}$, is Bertrand and Cournot rationalizable if and only if it satisfies ICC, MC and MDC.*

Proof. The result follows from combining the conditions in Theorems 1 and 2 and noting that a single-price data set trivially satisfies TDC. ■

The previous results started from the assumption that we are able to observe prices, outputs and cost information.[13] What happens to these results if we drop cost information from the data set? That is, can any set of generic homogeneous-good market observations be Bertrand or Cournot rationalized? If any set of generic observations of prices and outputs could be explained by the models then they can not be refuted even at a theoretical level. It turns out that there are non-trivial restrictions imposed by the Bertrand equilibrium upon generic data sets. To see this consider the following example of a symmetric duopoly, $n = 2$, with two observations, $m = 2$.

	(P, Q)
t=1	$(1, 4)$
t=2	$(2, 2)$

If these observations are Bertrand rationalizable then we should be able to find a cost function for the first observation which satisfies $0 < \bar{C}(4) \leq 4$ as firms do not make losses. In the second observation the monopoly deviation condition requires that we can choose the cost function so that $4 - \bar{C}(2) \geq 8 - \bar{C}(4)$ with $0 < \bar{C}(2) < \bar{C}(4)$. Inspection of these inequalities reveals that they cannot be simultaneously satisfied and the table is an example of a data set of prices and outputs which cannot be Bertrand rationalized. We now introduce two conditions for generic price and output data sets.

Definition 10 *A generic homogeneous-good set of prices and outputs, $(P_{it}, Q_{it})_{i \in N, t \in T}$, satisfies the revenue condition (RC) if, whenever $R_i(t) \neq \emptyset$, then $P_{t'}^* Q_{it'} > P_t^*(Q_t^* - Q_{it})$ for all $t' \in R_i(t)$ and $i \in A_t$.*

[13] We could have started instead by assuming that we observe prices, outputs and profits and then inferred cost information as the difference between revenue and costs.

The revenue condition states that the observed revenue which a firm obtains when it supplies an output at least as large as the observed aggregate output must be strictly greater than the increase in the revenue which a firm would obtain from supplying the entire market at the existing price.

Definition 11 *A generic homogeneous-good set of prices and outputs, $(P_{it}, Q_{it})_{i\in N, t\in T}$, satisfies the tied revenue condition (TRC) if $i \in A_t$ then we do not observe a $t' \in T$ such that $i \in N \setminus A_{t'}$ and $P^*_{t'}\hat{Q}_{t'} \geq P^*_t Q_{it}$ and $\hat{Q}_{t'} \leq Q_{it}$ with at least one of the inequalities holding strictly.*

The tied revenue condition requires that if we observe a firm supplying the market in a given observation then we should not observe the same firm not supplying the market whenever it can obtain at least as high a revenue and produce a strictly smaller output or produce the same output and obtain a strictly higher revenue. We now present the following result which shows that both the revenue condition and tied revenue condition are necessary conditions for Bertrand rationalizability.

Proposition 1 *A generic homogeneous-good set of prices and outputs, $(P_{it}, Q_{it})_{i\in N, t\in T}$, is Bertrand rationalizable only if it satisfies RC and TRC.*

Proof. Suppose we have a generic set of observations which violate RC. Then there is an i, t and $t' \in R_i(t)$ such that $P^*_{t'}Q_{it'} \leq P^*_t(Q^*_t - Q_{it})$. If the observations are rationalizable we should be able to find a cost function such that $\bar{C}_i(Q_{it'}) \leq P^*_{t'}Q_{it'}$ and $\bar{C}_i(Q_{it}) \leq P^*_t Q_{it}$ as firms do not make losses. MDC also requires that $\bar{C}_i(Q_{it'}) - \bar{C}_i(Q_{it}) > P^*_t(Q^*_t - Q_{it})$. However, if $\bar{C}_i(Q_{it'}) \leq P^*_{t'}Q_{it'}$ we must have that $\bar{C}_i(Q_{it'}) - \bar{C}_i(Q_{it}) < P^*_t(Q^*_t - Q_{it})$ and the observations are not Bertrand rationalizable.

Suppose that we have a set of generic observations which violate TRC. Then there is an i, t and t' such that $i \in A_t$, $i \in N \setminus A_{t'}$ and $P^*_{t'}\hat{Q}_{t'} > P^*_t Q_{it}$ and $\hat{Q}_{t'} \leq Q_{it}$. If the observations are rationalizable we should be able to find a cost function so that $\bar{C}_i(Q_{it}) \leq P^*_t Q_{it}$. However, as $\hat{Q}_{t'} \leq Q_{it}$ and $P^*_{t'}\hat{Q}_{t'} > P^*_t Q_{it}$ this implies that $P^*_{t'}\hat{Q}_{t'} - \bar{C}_i(\hat{Q}_{t'}) > 0$ and that firm i has a profitable deviation by posting the minimum price in observation t'.[14] ■

If one returns to the example data set it is clear that the reason that set of observations could not be Bertrand rationalized is because it violates RC: in observation one each firm produces a output equal to the aggregate market output in the second observation and obtains a revenue of 4. The revenue condition then requires that $4 > 2(4 - 2)$ which is violated. It is also straightforward to construct example data sets which satisfy the conditions

[14]The same argument applies when $P^*_{t'}\hat{Q}_{t'} \geq P^*_t Q_{it}$ and $\hat{Q}_{t'} < Q_{it}$.

in Proposition 1 and yet still fail to be Bertrand rationalizable so unfortunately the conditions cannot be easily extended to provide sufficient conditions. However, we shall introduce a strengthening of the revenue condition which can be applied to single-price observations of prices and outputs.

Definition 12 *A single-price data set of prices and outputs,* $(P_t^*, Q_{it})_{i \in N, t \in T}$*, satisfies the strengthened revenue condition (SRC) if, whenever* $R_i(t) \neq \emptyset$*, then* $P_{t'}^* Q_{it'} > P_t^* Q_t^*$ *for all* $t' \in R_i(t)$*.*

The following result shows that this condition alone is sufficient for a single-price set of prices and outputs to be Bertrand rationalizable.

Proposition 2 *A single-price data set of prices and outputs,* $(P_t^*, Q_{it})_{i \in N, t \in T}$*, is Bertrand rationalizable if it satisfies SRC.*

Proof. As SRC is satisfied we have $P_{r_i(t)}^* Q_{ir_i(t)} > P_t^* Q_t^*$ whenever this is defined.[15] Subtracting $P_t^* Q_{it}$ from each side of the inequality gives $P_{r_i(t)}^* Q_{ir_i(t)} - P_t^* Q_{it} > P_t^* Q_t^* - P_t^* Q_{it}$. The rationalizing cost function can then be chosen so that $\bar{C}_i(Q_{ir_i(t)}) = P_{r_i(t)}^* Q_{ir_i(t)}$, $\bar{C}_i(Q_{it}) = P_t^* Q_{it}$ and the constructed costs satisfy ICC. We then also have $\bar{C}_i(Q_{ir_i(t)}) - \bar{C}_i(Q_{it}) > P_t^*(Q_t^* - Q_{it})$ and MDC is satisfied. As the data set is a single-price data set TDC is trivially satisfied and the rationalizability of the data set follows from Theorem 1. ■

Up until now we have not considered what structure the Cournot equilibrium imposes upon single-price data sets of prices and outputs and what the relationship is with the Bertrand equilibrium conditions. It turns out that the Cournot equilibrium imposes *no* restrictions upon prices and outputs. The following result states that *any* single-price data set of prices and outputs can be Cournot rationalized.

Proposition 3 *Any single-price data set of prices and outputs,* $(P_t^*, Q_{it})_{i \in N, t \in T}$*, is Cournot rationalizable.*

Proof. See Carvajal and Quah (2009, p.19, Corollary 1). ■

This represents a significant difference between the two models. If one permits a general increasing cost function then any single-price data set of prices and outputs can be Cournot rationalized whereas the Bertrand equilibrium imposes non-trivial restrictions even upon single-price data sets. Once cost information is unavailable the Cournot model cannot be refuted. However, if one restricts the rationalizing cost function beyond requiring that it just explain the observed costs and be strictly increasing then it is possible that some single-price data sets of prices and outputs cannot be Cournot rationalized.[16]

[15] Recall that $r_i(t) = \{t' \in R_i(t) : Q_{it'} < Q_{it} \, \forall t \in R_i(t)\}$ was introduced in the proof of Theorem 1.

[16] Carvajal and Quah (2009) introduce the notion of a 'convincing rationalization' where the cost function

3 Examples of revealed Nash equilibria in oligopoly games

One of the advantages of the revealed preference method is that we can write down possible observations and test their compatibility with different equilibrium concepts. We now proceed in this fashion and apply the conditions derived in the previous section to two simple data sets to demonstrate their theoretical usefulness.

3.1 Example 1

	(P, Q, C)
t=1	$(3, 1, 1)$
t=2	$(4, 2, 7)$

Consider the example observations given in the table of a symmetric duopoly, $n = 2$, with two observations, $m = 2$. This is a single-price data set and could potentially be consistent with either the Bertrand or Cournot equilibrium solutions. First, we can note that $S_i(2) = \{1\}$. The increasing cost condition requires $C_{i2} - C_{i1} = 7 - 1 > 0$ which is satisfied. Second, note that $R_i(1) = \{2\}$ as in observation two each firm produces an output equal to the aggregate market output in observation one. The monopoly deviation condition requires that we do not observe a firm able to increase its observed profits by supplying the entire market demand. This means we require $P_{i1}^* Q_{i1} - C_{i1} = (3)(1) - (1) = 2 \geq (3)(2) - 7 = -1 = P_1^* Q_1^* - C_{i2}$ which is satisfied. As this is a single-price data set the tie deviation condition is trivially satisfied and we can conclude that the observations can be Bertrand rationalized. Now consider the marginal condition. We require that we do not observe a situation where a firm could benefit from reducing its output. This means we require $P_2^* Q_{i2} - C_{i2} = (4)(2) - 7 = 1 > (4)(1) - 1 = 3 = P_2^* Q_{i1} - C_{i1}$ which is violated. We can conclude that this set of observations *cannot* be Cournot rationalized.

3.2 Example 2

Consider the example observations given in the table above of a symmetric duopoly, $n = 2$, and two observations, $m = 2$. Again as this is a single-price data set the observations could

must be chosen so that the marginal cost lies between the observed marginal costs and show that this restriction rules out certain types of observations being Cournot rationalizable.

	(P,Q,C)
t=1	$(3,1,2)$
t=2	$(4,2,4)$

potentially be consistent with firms playing either the Bertrand or Cournot equilibrium. First, note that $S_i(2)=\{1\}$. The increasing cost condition requires $C_{i2}-C_{i1}=4-2>0$ which is satisfied. Second, note that $R_i(1)=\{2\}$ as in observation two each firm produces an output equal to the aggregate output in the first observation. The monopoly deviation condition requires that $P^*_{i1}Q_{i1}-C_{i1}=(3)(1)-2=1\geq(3)(2)-4=2=P^*_1Q^*_1-C_{i2}$ which is violated. Therefore this set of observations *cannot* be Bertrand rationalized. The marginal condition requires that no firm can benefit by reducing their outputs. The condition requires that $P^*_2Q_{i2}-C_{i2}=(4)(2)-4=4>(4)(1)-2=2=P^*_2Q_{i1}-C_{i1}$. Therefore this set of observations can be Cournot rationalized.

4 Conclusion

Following on from the works of Sprumont (2000), Zhou (2005) and Carvajal and Quah (2009) which characterized Nash equilibrium sets in abstract games, the aim of this paper has been to show that the method of revealed preferences can be applied to oligopoly games. To this end, we have provided characterizations of observations which can be rationalized by the Bertrand and Cournot equilibrium solutions. The results show that both equilibrium concepts impose non-trivial restrictions upon observables and it is possible for a set of market observations to be revealed inconsistent with either of the equilibrium concepts. Moreover, the conditions which characterize the sets, monopoly deviation, tie deviation and marginal condition, are economically intuitive and take the form of linear inequalities. However, if we only observe prices and outputs then the Cournot equilibrium imposes no restrictions upon the observables whereas the Bertrand equilibrium does. An open problem is what restrictions other oligopoly models impose upon observable outcomes. For example, in Bertrand-Edgeworth competition where firms are free to ration the demand which they receive, then a homogeneous-good may be traded at different prices, which is true of real-world markets, and the market outcomes may be quite different from Bertrand or Cournot competition. The difficulty in proceeding in this direction is that the equilibrium solution is often in mixed strategies and it is not straightforward how revealed preference theory should be extended to test stochastic choice behaviour.[17]

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