SOVEREIGN DEBT AND DEFAULT INCENTIVES UNDER INCOMPLETE MARKETS

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Abstract

This paper revisits the question of whether a sovereign's loss of reputation provides by itself strong incentives for repayment when financial markets offer incomplete insurance opportunities. Reputational debt might be enforceable, as apposed to the Impossibility Theorem of Bulow and Rogoff (1989), unless relevant conditions are imposed on the pricing kernel. We present examples in which positive borrowing is sustained even though the sovereign's natural debt limits, corresponding to the most pessimistic evaluation of future endowment, are finite. In general, no-sustainability of reputational debt requires a stronger version of high implied interest rates, namely, the value of the most optimistic evaluation of future endowment be finite.

Keywords: Sovereign risk, Ponzi games, Reputational debt, Incomplete markets.

JEL Classification: F34, H63.

1. Introduction

The renowned Impossibility Theorem by Bulow and Rogoff (1989) asserts that sovereign debt is unsustainable if debt contracts are not supported by direct sanctions and default induces only a ban from ever borrowing in financial markets.[1] The intuition is that, when solvency constraints preclude the roll-over of debt, a country can always improve upon contractual arrangements that involve repayments (*i.e.*, positive net transfers from the country

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[1]Bulow and Rogoff (1989) led to a vast literature studying alternative mechanisms to answer why countries repay their debts in the absence of sanctions. We refer to Aguiar and Amador (2014) and Wright (2011) for a thorough discussion of the literature.

to foreign investors) by defaulting at the contingency associated with the maximum expansion of debt. When markets are complete, those solvency conditions require high implied interest rates (*i.e.*, equilibrium prices that imply a finite present value of future aggregate endowment), for otherwise the sovereign can roll-over existing debt indefinitely.[2]

The logic underlying the unsustainability of reputational debt is deeper than intuition suggests (and, as a matter of fact, its proof is subtler).[3] Even though solvency constraints impose debt redemption, repayments might be extremely dispersed over time under uncertainty. During the repayment phase, trade in financial instruments is still necessary for protection against adverse shocks, so that expanding liabilities is indispensable for consumption smoothing purposes. For default be profitable because of saved repayments, financial markets must be allowing for a similar risk diversification when borrowing is prohibited. Under incomplete markets, it is not clear whether such replication strategy might be feasible. This casts some doubts on the validity of Bulow and Rogoff (1989)'s Impossibility Theorem. The purpose of this paper is to provide a better understanding of this issue.

We provide examples of sovereign debt sustained by reputation under incomplete markets. The explanation for stronger incentives to debt repayment is that some insurance opportunities cannot be replicated after default, as liabilities are inhibited. Default involves a benefit (saved repayments) along with a cost (severe incomplete risk-diversification). The incentive to default depends on this trade-off and, for high risk-aversion, the cost may overcome the benefit. This suggests that, in order to restore the validity of the Impossibility Theorem, we need to identify conditions under which replication is feasible under incomplete markets.

When markets are incomplete, evaluation is ambiguous on non-tradable claims. At a competitive equilibrium, under full commitment, debt is restricted by feasible repayment, the natural debt limit, corresponding to the most pessimistic evaluation of future endowment. Our examples reveal that the Impossibility Theorem might fail even though natural debt limits are finite, a legitimate analog of high implied interest rates for incomplete markets. We argue that, in general, replication requires that the most optimistic evaluation of future endowment be also finite, a substantially stronger condition. The reason is that, as in Santos and Woodford (1997), this guarantees the existence of a trading strategy that finances any budget-affordable net consumption plan through portfolios involving no liabilities. We prove

[2]Hellwig and Lorenzoni (2009) (see also Bidian and Bejan (2014)) show that low implied interest rates is a necessary and sufficient condition for the exact roll-overing of equilibrium self-enforcing debt limits.

[3]Martins-da-Rocha and Vailakis (2014) show that the original argument in Bulow and Rogoff (1989) does not go through when the output of the sovereign may vanish along a path of successive low productivity shocks, or when it may grow unboundedly along a path of successive high productivity shocks.

that this is sufficient to restore the validity of the Impossibility Theorem under an additional mild technical assumption.

Pesendorfer (1992) studies repayment incentives of small open economies trading with competitive, risk-neutral foreign investors while having access to a limited set of financial assets. This work differentiates from ours in a very crucial aspect. The punishment in Pesendorfer (1992) is that defaulters are precluded from holding a negative position in each of the available assets. Such a strong default consequence supports positive borrowing even if there is a complete set of assets after default, provided that debtors cannot form portfolios consistent with full insurance. We instead prevent defaulters to buy portfolios that involve liabilities. This is a weaker requirement that is much closer in spirit to the cash-in-advance contracts of Bulow and Rogoff (1989). When a sovereign reneges on a payment, it might negotiate with an intermediary a sequence of up-front payments in exchange of positive contingent deliveries in the future. The intermediary will form portfolios of available securities to meet its obligations, with cash-in-advance payments reflecting their market value, while the sovereign will never be in the condition of defaulting on this arrangement. It is irrelevant whether the portfolios are formed by taking negative positions on some assets.

The paper is organised as follows. In Section 2 we present examples that deliver the economic insight underlying the failure of Bulow and Rogoff (1989)'s unsustainable debt result in incomplete markets. In Sections 3 and 4 we lay out the fundamentals of the economy and we discuss the meaning as well as the link of various restrictions on asset pricing kernel. Section 5 shows which conditions on arbitrage-free prices restore the validity of Bulow and Rogoff (1989)'s Impossibility Theorem. The formal proof is presented in Appendix A. It is worth noticing that we elaborate on an innovative approach that applies independently of the extension to incomplete markets. In that respect, the original result in Bulow and Rogoff (1989) and the extension proposed in Martins-da-Rocha and Vailakis (2014) are both derived as by-products of our analysis. For completeness, some technical properties of incomplete-markets pricing are presented in Appendix B.

2. Examples

We here present some examples of failure of Bulow and Rogoff (1989)'s Impossibility Theorem under incomplete markets. The cause of this failure is that the incompleteness of markets does not allow for replication when liabilities are prohibited after default. This implies that, in general, a country cannot benefit from defaulting and, hence, sovereign debt is sustainable by reputation.

Example 2.1. The first example is simple but it delivers the basic intuition underlying the failure of Bulow and Rogoff (1989)'s unsustainable debt result. The economy is subject to binomial uncertainty over states $\{D, U\}$ occurring with equal probability. Markets are incomplete, as there is a single asset with payoffs $(y_D, y_U) = (1, -1)$ and price $q = 0$. The endowment is $(e_D, e_U) = (0, 2)$. The economy begins with state U and an initial liability $v = -1$. Trivially, these conditions permit complete insurance at constant consumption $c = 1$. Whenever in state U, the country holds a liability $v = -1$. We ask whether default is profitable in this setting.

Upon default, liabilities are not allowed. Hence, in this simple economy, no asset can be traded and autarchy is the only budget-feasible consumption after default. In general, this cannot be a benefit, thus violating Bulow and Rogoff (1989)'s Impossibility Theorem. Indeed, if intertemporal preferences are additively separable with $1 > \beta > 0$ be the discount factor, a sufficient condition is that the instantaneous utility function satisfies the following inequality

$$u(2) - u(1) < \frac{\beta}{1-\beta}\Big[u(1) - \frac{u(2) + u(0)}{2}\Big].$$

That is, the sovereign cannot benefit from defaulting if the current gain of not repaying his debt is compensated by the gain of smoothing future consumption.

Example 2.2. Here is a more complicated example, as the asset structure now permits a (strictly) positive transfer. Uncertainty is given by states $\{D, M, U\}$ with transition probabilities $(1/3, 1/3, 1/3)$. There are only two securities. For some $\epsilon > 0$, one security is paying off $(y_D, y_M, y_U) = (1, \epsilon, 0)$ while the other one is paying off $(y_D, y_M, y_U) = (0, \epsilon, 1)$. The price of each security is $q = (1/3)\,\beta\,(1 + \epsilon)$, where $1 > \beta > 0$ is the discount factor. The endowment is $(e_D, e_M, e_U) = (0, 1, 2)$. The initial state is U with an initial liability $v = -1$.

It is easy to see that the portfolio with deliveries $(v_D, v_M, v_U) = (1, 0, -1)$ sustains full-insurance at constant consumption $c = 1$. Furthermore, it is optimal for all bounded debt limits permitting the liability $v = -1$ in state U and no liabilities in other states. We now argue that, in these circumstances, there are robust cases of a violation of Bulow and Rogoff (1989)'s Impossibility Theorem.

Consider a utility function of the form

$$u\,(c) = \frac{c^{1-(1/\epsilon)} - 1}{1 - (1/\epsilon)},$$

where $\epsilon > 0$ is the elasticity of intertemporal substitution. Notice that, for any ϵ in the interval $(0, 1/2)$, this utility is uniformly bounded from above, as

$$(*) \qquad u(c) \leq \frac{-1}{1-(1/\epsilon)} \leq 1.$$

Furthermore, as it can be verified by direct computation, for every $1 > \eta > 0$,

$$(**) \qquad \lim_{\epsilon \to 0} u(1-\eta) = \lim_{\epsilon \to 0} \frac{(1-\eta)^{1-(1/\epsilon)} - 1}{1-(1/\epsilon)} = -\infty.$$

Does the country benefit from defaulting in state U, when holding a liability? To answer this, we evaluate default incentives along a sequence of monotonically vanishing $\epsilon > 0$. Indeed, suppose that there exists a sequence of consumption plans $(c^\epsilon)_{\epsilon>0}$ such that each plan is supported by a trading strategy involving no liabilities and guarantees an overall utility, after defaulting in state U, at least equal to the over-all utility from full insurance, that is,

$$V_U(c^\epsilon) \geq 0.$$

Budget feasibility with no liabilities imposes that, if consumption is bounded by $\xi_t > 0$ in period t, it is bounded by $\xi_{t+1} > 0$ in the following period, where

$$\frac{\xi_t}{(1/3)\beta} \leq \xi_{t+1}.$$

Hence, at no loss of generality, by Tychonoff's Theorem, it can be assumed that the sequence of consumption plans converges and that, in every period t, at every contingency, $c_t^0 = \lim_{\epsilon \to 0} c_t^\epsilon \leq \xi_t$. Suppose that, at some contingency, $c_t^0 < 1 - \eta$ for some $1 > \eta > 0$. By condition (**), this implies an infinite loss, which cannot be compensated by bounded gains in other periods, because of (*). Hence, at every contingency, the consumption in the limit exceeds the full-insurance consumption, that is, $c_t^0 \geq 1$. We now argue that this yields a contradiction, thus proving that, for every sufficiently small $\epsilon > 0$, there is not benefit from default in the only state in which the country holds a liability.

In period t, the purchase of each asset is bounded by $\xi_{t+1} > 0$ and, hence, the delivery of the portfolio in the following period, conditional on state M, is bounded by $\epsilon\xi_{t+1}$. As a consequence, in the limit, no resources are transferred from the previous period in state M, whenever it occurs. Beginning from state M, the consumption converges to the full-insurance value, so that no resources are invested in the limit. Therefore, no additional resources are available for consumption in the following period when the current state is D, which makes it impossible to guarantee the full-insurance consumption.

3. Markets and prices

3.1. **Uncertainty.** The economy extends over an infinite horizon, $\mathbb{T} = \{0, 1, 2, 3, \ldots, t, \ldots\}$, subject to uncertainty. Uncertainty is represented by a probability space, $(\Omega, \mathcal{F}, \mu)$, and a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ of σ-algebras. To simplify, and to avoid issues of integrability, it is assumed that $\mathcal{F}_0$ is the trivial σ-algebra and, for every t in $\mathbb{T}$, $\mathcal{F}_t$ is a σ-algebra generated by a *finite* partition of Ω. Given a state of nature ω in Ω, at every period t in $\mathbb{T}$, $\mu(\mathcal{F}_t(\omega)) > 0$, where $\mathcal{F}_t(\omega) = \cap \{E_t \in \mathcal{F}_t : \omega \in E_t\}$ represents the available information. Such primitive events are referred to as contingencies. In the equivalent event-tree representation of uncertainty, this corresponds to a date-event.

3.2. **Linear spaces.** The linear space L consists of all maps $f : \mathbb{T} \times \Omega \to \mathbb{R}$ such that, for every t in $\mathbb{T}$, $f_t : \Omega \to \mathbb{R}$ is $\mathcal{F}_t$-measurable, an element of the linear space L_t. The linear space L decomposes as $L = \oplus_{t\in\mathbb{T}} L_t$. An adapted process f in L is positive whenever, at every t in $\mathbb{T}$, $f_t(\omega) \geq 0$ for all ω in Ω. As usual, $f \geq 0$ denotes positivity and $f > 0$ non-null positivity. Strict positivity corresponds to the case in which, at every t in $\mathbb{T}$, $f_t(\omega) > 0$ for every ω in Ω. The positive cone of L is denoted by L^+.

3.3. **Tradable claims.** Incomplete markets are represented by a linear subspace $V = \oplus_{t\in\mathbb{T}} V_t$ of $L = \oplus_{t\in\mathbb{T}} L_t$. In other terms, V is interpreted as the space of tradable contingent claims. As markets are sequential, we assume that, at every t in $\mathbb{T}$, v_{t+1} lies in V_{t+1} only if $v_{t+1}\chi_{E_t}$ is also in V_{t+1} for every event E_t in $\mathcal{F}_t$, where χ_E is the indicator function for every event E in $\mathcal{F}$. We maintain the assumption that some strictly positive element u on L is also in V, that is, available financial instruments allows for a (possibly risky) strictly positive transfer. The presence of a safe asset would be sufficient, though it is more demanding than necessary.

3.4. **Pricing.** In every period t in $\mathbb{T}$, the asset pricing kernel is given by a linear mapping $\varphi_t : V_{t+1} \to L_t$. The interpretation is that $\varphi_t(v_{t+1})$ is the market price of any portfolio with deliveries v_{t+1} in V_{t+1}. No arbitrage implies that, whenever v_{t+1} is a non-null claim in $V_{t+1} \cap L^+_{t+1}$, then $\varphi_t(v_{t+1}) > 0$. In other terms, any positive claim is costly on the market.

An (implicit) price p is a *strictly positive* element of L satisfying, at every t in $\mathbb{T}$,

$$\varphi_t(v_{t+1}) = \frac{1}{p_t}\mathbb{E}_t p_{t+1} v_{t+1}.$$

By the assumption of no arbitrage, along with a trivial application of Riesz Representation Theorem, implicit prices exist and form a (non-empty) convex cone P. This provides an equivalent representation of the asset pricing kernel. Indeed, as prices are invariant on the space of tradable claims V, in every period t in $\mathbb{T}$, the market value of claim v_{t+1} in V_{t+1} is given by

$$\bigwedge_{p\in P} \frac{1}{p_t}\mathbb{E}_t p_{t+1} v_{t+1} = \bigvee_{p\in P} \frac{1}{p_t}\mathbb{E}_t p_{t+1} v_{t+1}.$$

Here we take these basic facts as a primitive framework. They are well-established results in the literature and need no justification. To make the paper self-contained, the relevant theorems are collected in Appendix B.

3.5. **Ponzi games.** A country is running a Ponzi scheme whenever it is persistently financing some liability by means of a new liability. In other terms, when a liability is rolled over indefinitely. We now precisely identify circumstances under which such a Ponzi game occurs.

A financial plan v in V involves a Ponzi scheme if there exists a non-null b in L^+ such that, at every t in $\mathbb{T}$,

(PG-1) $$b_t \leq v_t^-$$

and

(PG-2) $$b_t \leq \bigwedge_{p\in P} \frac{1}{p_t}\mathbb{E}_t p_{t+1} b_{t+1},$$

where we use the canonical decomposition $v = v^+ - v^-$, separating claims from liabilities.

An important observation is that, even if the plan b does not belong on the space of tradable claims V, by the Fundamental Theorem of Duality (see Appendix B), there always exists a tradable plan b^* in V such that, at every t in $\mathbb{T}$,

$$b_t^* \leq b_t$$

and

$$b_t^* \leq \bigwedge_{p\in P} \frac{1}{p_t}\mathbb{E}_t p_{t+1} b_{t+1}^* = \bigwedge_{p\in P} \frac{1}{p_t}\mathbb{E}_t p_{t+1} b_{t+1}.$$

Hence, some outstanding liabilities are served by new liabilities and, thus, are rolled over indefinitely.

4. High interest rates

Under full commitment, liabilities are traditionally bounded by natural debt limits. This is justified by the requirement that sustainable debts need be repayable in finite time out of the endowment. As established by Hernández and Santos (1996), Levine and Zame (1996) and Santos and Woodford (1997), when markets are incomplete, the natural debt limit is given by the worst evaluation of future endowment that needs to be finite. Thus, at a competitive equilibrium of this sort,

$$\bigwedge_{p\in P}\frac{1}{p_0}\mathbb{E}_0\sum_{t\in\mathbb{T}}p_te_t \text{ is finite,}$$

where e in L^+ is the adapted process representing the endowment.

Our examples show that, in general, the Impossibility Theorem fails under this restriction, because replication might be unfeasible after default. We need to further restrict prices by assuming that

$$\text{(F)}\qquad \bigvee_{p\in P}\frac{1}{p_0}\mathbb{E}_0\sum_{t\in\mathbb{T}}p_te_t \text{ is finite.}$$

That is, the value of the endowment is (uniformly) finite for all prices consistent with the absence of arbitrage opportunities. In analogy with the terminology used in complete markets (see Alvarez and Jermann (2000)), we refer to this property as *high implied interest rates.*

Under complete markets, high implied interest rates deliver a property of continuity of the pricing kernel in a topology which is coherent with impatience: the value of residual claims in the remote future vanishes. We need a similar property under incomplete markets, namely,

$$\text{(H)}\qquad \lim_{t\to\infty}\bigvee_{p\in P}\frac{1}{p_0}\mathbb{E}_0\sum_{s\in\mathbb{T}}p_{t+s}e_{t+s}=0.$$

When the pricing kernel satisfies condition (H) we say that it exhibits *uniformly high implied interest rates.*

Remark 4.1. It is easy to see that condition (H) implies condition (F). Indeed, assuming that (F) is violated, we can show that, for every $\epsilon>0$, at every t in $\mathbb{T}$,

$$\bigvee_{p\in P}\frac{1}{p_0}\mathbb{E}_0\sum_{s\in\mathbb{T}}p_{t+s}e_{t+s}\geq\epsilon,$$

thus violating restriction (H). To this purpose, it suffices to argue that, for every t in $\mathbb{T}$,

$$\bigvee_{p\in P}\frac{1}{p_0}\mathbb{E}_0\sum_{s=0}^{t}p_se_s \text{ is finite.}$$

This is what we accomplish in the following, by noticing that some strictly positive claim u is in the tradable space V.

Peg any t in $\mathbb{T}$ and suppose that w_{t+1} in $V_{t+1} \cap L_{t+1}^{+}$ is given. It is immediate to verify that there exists some w_t in $V_t \cap L_t^{+}$ such that, for every price p in P,

$$\mathbb{E}_t p_{t+1} w_{t+1} + p_t e_t \leq p_t w_t.$$

This is true because, for every $\lambda > 0$, the expansion λu_t is an arbitrarily large strictly positive claim in $V_t \cap L_t^{+}$. Hence, by setting $w_{t+1} = 0$ and proceeding by backward induction, it follows that, for every price p in P,

$$\mathbb{E}_0 \sum_{s=0}^{t} p_s e_s \leq p_0 w_0,$$

thus proving the claim.

Remark 4.2. Here is an important case in which condition (H) is satisfied. Let us consider an economy with a tradable non-contingent bond in which there exists a sufficiently large $1 > \beta > 0$ such that, at every t in $\mathbb{T}$,

$$\beta > \bigvee_{p \in P} \frac{1}{p_t} \mathbb{E}_t p_{t+1}.$$

As the right-hand side is the price of the bond, this restriction imposes a sort of lower bound on the interest rates uniformly across all contingencies. It is straightforward to verify that, whenever this uniform lower bound exists, the hypothesis of uniformly high implied interest rates is satisfied in an economy with *bounded* endowment. Indeed,

$$\bigvee_{p \in P} \frac{1}{p_0} \mathbb{E}_0 \sum_{s \in \mathbb{T}} p_{t+s} e_{t+s} \leq \sum_{s \in \mathbb{T}} \beta^{t+s} \|e\|_\infty \leq \beta^t \left(\frac{1}{1-\beta} \right) \|e\|_\infty .$$

Remark 4.3. We here present a relevant case in which *high implied interest rates are violated* when the non-contingent bond is the only tradable asset. Suppose that the price of this bond is constantly equal to $1 > \beta > 0$ and that the endowment evolves as a random walk. In such environment, a particular price is given by $p_t = \beta^t e_t$, at every t in $\mathbb{T}$, because

$$\frac{1}{p_t} \mathbb{E}_t p_{t+1} = \beta \frac{1}{e_t} \mathbb{E}_t e_{t+1} = \beta.$$

It seems not an extremely severe restriction to postulate that

$$\frac{1-\beta}{\beta} < \operatorname{var}_t \left(\frac{e_{t+1}}{e_t} \right),$$

where var_t (cov_t) denotes the variance (covariance) conditional on information available at t in $\mathbb{T}$. It is easy to verify that that

$$1 < \mathbb{E}_t \left(\frac{p_{t+1}}{p_t} \right) \mathbb{E}_t \left(\frac{e_{t+1}}{e_t} \right) + \text{cov}_t \left(\frac{p_{t+1}}{p_t}, \frac{e_{t+1}}{e_t} \right).$$

Thus, high implied interest rates are fatally violated, as

$$1 < \bigvee_{p \in P} \frac{1}{p_t e_t} \mathbb{E}_t p_{t+1} e_{t+1}.$$

5. Default incentives

We here show that, under uniformly high implied interest rates, sovereign debt is unsustainable. This extends Bulow and Rogoff (1989)'s Impossibility Theorem (see also Martins-da-Rocha and Vailakis (2014)) under more restrictive assumptions than those for complete markets. Importantly, we provide an alternative argument which applies independently of the extension of market incompleteness.

A contract c in L^+ is budget-feasible if there exists a financial plan v in V, involving no Ponzi scheme, such that, at every t in $\mathbb{T}$,

$$\mathbb{E}_t p_{t+1} v_{t+1} + p_t (c_t - e_t) \leq p_t v_t.$$

Budget restrictions are compatible with liabilities, though debt cannot be rolled over indefinitely. Ruling out Ponzi games is equivalent to bound liabilities by the most favourable evaluation of future endowment. Notice that, in general, this is more permissive that the bound given by the natural debt limit, that is, the most pessimistic evaluation of future endowment (see Santos and Woodford (1997)).

Proposition 5.1 (Bounds to liabilities)**.** Under high implied interest rates, a contract c in L^+ is budget-feasible only if its underlying financial plan v in V satisfies, at every t in $\mathbb{T}$,

$$v_t + g_t \geq 0,$$

where the adapted process g in L^+ is given by

$$g_t = \bigvee_{p \in P} \frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathbb{T}} p_{t+s} e_{t+s}.$$

A contract c in L^+ is budget-feasible *with no liabilities* if there exists a financial plan w in $V \cap L^+$ such that, at every t in $\mathbb{T}$,

$$\mathbb{E}_t p_{t+1} w_{t+1} + p_t (c_t - e_t) \leq p_t w_t.$$

Let B_t be the set of all such budget-feasible contracts c in L^+ which involve no claims (*i.e.*, $w_t = 0$) in period t in $\mathbb{T}$. A contract c in L^+ is immune to default if, at every t in $\mathbb{T}$,

$$U_t(c) \geq \sup_{c^* \in B_t} U_t(c^*),$$

where $U_t : L^+ \to L_t$ is the strictly monotone utility, conditional on relevant information, beginning from period t in $\mathbb{T}$. In other terms, a contract is immune to default whenever, at every contingency, a country would not benefit from defaulting and trading subject to no borrowing in the future.

We now restore the Impossibility Theorem under uniformly high implied interest rates. The hypothesis of high implied interest rates guarantees that replication is feasible. The default option is profitable unless the sovereign can roll-over debt over time. Such Ponzi games are ruled out by assuming continuity of the pricing kernel (i.e., assuming uniformly high implied interest rates).

Proposition 5.2 (Sovereign debt paradox)**.** Under uniformly high implied interest rates, a budget-feasible contract c in L^+ is immune from default if and only if it involves no liabilities, that is, any financial plan supporting this contract is positive.

The intuition for this result is completely exhausted by the deterministic case, once one takes an approach slightly different from Bulow and Rogoff (1989)'s original argument. In the deterministic case, the budget constraint imposes, at every t in $\mathbb{T}$,

$$p_{t+1}v_{t+1} + p_t(c_t - e_t) \leq p_t v_t.$$

Define b_t as the supremum over the present value of liabilities across all possible truncations, that is,

$$b_t = \bigvee_{\tau \in \mathbb{T}} \frac{1}{p_t} p_{t+\tau}(-v_{t+\tau}) \geq -v_t.$$

When there is uncertainty, such truncations need be contingent and, when markets are incomplete, one needs to consider the largest evaluation of future liabilities. Under incomplete markets, the largest evaluation permits to recover a plan for tradable contingent claims by means of the Fundamental Theorem of Duality.

It is immediate to see that $p_t b_t = \max\{-p_t v_t,\ p_{t+1}b_{t+1}\}$, so the process is obviously a super-martingale,

$$p_t b_t \geq p_{t+1} b_{t+1}.$$

This crucial observation reveals that replication is feasible, as the financial plan $v_t + b_t \geq 0$ involves no liabilities and

$$p_{t+1}\left(v_{t+1} + b_{t+1}\right) + p_t\left(c_t - e_t\right) \leq p_t\left(v_t + b_t\right)$$

Moreover, when the inequality is slack,

$$p_t b_t > p_{t+1} b_{t+1},$$

then $w_t = v_t + b_t = 0$, which uncovers a benefit from defaulting (and restarting with $w_t = 0$). Intuitively, in this situation, sovereign debt has reached its maximum expansion and, thus, the country begins a repayment policy. However, defaulting allows the country to save on these repayments and enjoy higher consumption. Thus, when defaulting is not profitable, the adapted process is indeed a martingale, that is, at every t in $\mathbb{T}$,

$$p_t b_t = p_{t+1} b_{t+1}.$$

This basically means that sovereign debt is persistently expanding over time and, therefore, a repayment policy never begins. This sort of Ponzi games is ruled out by the hypothesis of uniformly high implied interest rates.

6. Conclusion

We have shown, by means of examples, that market incompleteness may induce strong incentives for repayment when liabilities are prohibited after default. A sovereign may not benefit from defaulting on its debt and positive borrowing can be sustainable by reputation. To restore the Bulow and Rogoff (1989)'s impossibility result, one needs to impose strong restrictions on the pricing functional. In particular, replication is obtained under a continuity property implying that the value of the most optimistic evaluation of future endowment eventually vanishes in the long-run.

Appendix A. Proofs

Proof of proposition 5.1. By construction of the adapted process g in L^+, at every t in $\mathbb{T}$, for every price p in P,

$$\mathbb{E}_t p_{t+1} g_{t+1} + p_t e_t \leq p_t g_t.$$

Hence, adding up with the budget constraints, for every price p in P,

$$\mathbb{E}_t p_{t+1}\left(v_{t+1} + g_{t+1}\right) \leq p_t\left(v_t + g_t\right).$$

In turn, this implies that, for every price p in P,

$$p_t (v_t + g_t)^- \leq \mathbb{E}_t p_{t+1} (v_{t+1} + g_{t+1})^- .$$

Finally, at every t in $\mathbb{T}$,

$$v_t^- - (v_t + g_t)^- \geq 0.$$

Therefore, as Ponzi schemes are ruled out, we conclude that $v+g \geq 0$, for otherwise conditions (PG-1)-(PG-2) would be satisfied by $b = (v+g)^-$ in L^+. □

Proof of proposition 5.2. At every t in $\mathbb{T}$, define $\mathcal{T}_t$ as the set of all finite-time contingent truncations (which includes $\tau = 0$). This is the space of all maps $\tau : \Omega \to \mathbb{T}$ such that $\tau \leq n$, for some n in $\mathbb{T}$, and $\{\omega \in \Omega : \tau(\omega) = n\}$ belongs to $\mathcal{F}_{t+n}$, for every n in $\mathbb{T}$. As usual, for an element x of L, given a contingent truncation τ in $\mathcal{T}_t$, $x_{t+\tau} : \Omega \to \mathbb{R}$ is the $\mathcal{F}$-measurable map defined by $x_{t+\tau}(\omega) = x_{t+\tau(\omega)}(\omega)$.

Consider the adapted process b in L obtained, at every t in $\mathbb{T}$, by

$$b_t = \bigvee_{p \in P} \bigvee_{\tau \in \mathcal{T}_t} \frac{1}{p_t} \mathbb{E}_t p_{t+\tau} (-v_{t+\tau}) \leq \bigvee_{p \in P} \bigvee_{\tau \in \mathcal{T}_t} \frac{1}{p_t} \mathbb{E}_t p_{t+\tau} (g_{t+\tau}) \leq g_t,$$

where the first inequality is due to Proposition 5.1. It is immediate to verify that $v_t + b_t \geq 0$ and

$$b_t \geq \bigvee_{p \in P} \frac{1}{p_t} \mathbb{E}_t p_{t+1} b_{t+1}.$$

This inequality is obvious once one observes that

$$\{\tau + 1 : \tau \in \mathcal{T}_{t+1}\} \subset \mathcal{T}_t.$$

Furthermore, by the Fundamental Theorem of Duality (see Appendix B), there exists a tradable claim b^* in V such that, at every t in $\mathbb{T}$,

$$b_t^* \geq b_t$$

and

$$b_t^* \geq \bigvee_{p \in P} \frac{1}{p_t} \mathbb{E}_t p_{t+1} b_{t+1}^* = \bigvee_{p \in P} \frac{1}{p_t} \mathbb{E}_t p_{t+1} b_{t+1}.$$

Suppose that, at some contingency,

$$b_t > \bigvee_{p \in P} \frac{1}{p_t} \mathbb{E}_t p_{t+1} b_{t+1} \text{ only if } v_t + b_t = 0.$$

We show that this delivers an immediate contradiction. Indeed, consider the financial plan w in V that satisfies $w_t = v_t + b_t = 0$ and $w_{t+s} = v_{t+s} + b^*_{t+s} \geq v_{t+s} + b_{t+s} \geq 0$ for every s in $\mathbb{N}$. This plan shows that, in period t in $\mathbb{T}$, the consumption plan c is budget-feasible with no liabilities, an element of B_t, and it does not exhaust resources, a contradiction as utility is strictly monotone. Hence,

$$(*) \qquad b_t = \bigvee_{p \in P} \frac{1}{p_t} \mathbb{E}_t p_{t+1} b_{t+1}.$$

Assume that $b_0 > 0$. By (*), there exists a price p in the closure of P, with $p_0 > 0$, such that, at every t in $\mathbb{T}$,

$$p_t b_t \leq \mathbb{E}_t p_{t+1} b_{t+1}.$$

Hence, observing that Ponzi schemes are excluded,

$$p_0 b_0 \leq \mathbb{E}_0 p_t b_t \leq \mathbb{E}_0 p_t g_t.$$

Taking the limit, by uniformly high implied interest rates, this implies that $0 < b_0 \leq 0$, a contradiction. Reproducing the argument beginning from any contingency, $b \leq 0$ and, as $v + b \geq 0$, $v \geq 0$, thus proving the claim. □

Appendix B. Arbitrage-free pricing

We here collect some basic facts about arbitrage-free asset pricing which are used in the body of the text. These are well-known implications of the Fundamental Theorem of Duality. We provide simple proofs for convenience, independently of their applications in this paper.

The space of tradable claims Y is a linear subspace of some (finite-dimensional) linear space X, endowed with its canonical ordering. The pricing of tradable claims is given by a linear map $\varphi : Y \to \mathbb{R}$. This map is arbitrage free, in the sense that, for any claim y in Y, $y > 0$ only if $\varphi(y) > 0$. We assume that there exists a strictly positive tradable claim u in Y with $\varphi(u) = 1$. This needs not be the safe asset, though a safe asset would be sufficient for this property be satisfied. As X is an Hilbert space, the internal product is denoted by $x \cdot y$. Let Π be the convex set of positive linear functionals π in X such that, for every y in Y,

$$\varphi(y) = \pi \cdot y.$$

Here is the Fundamental Theorem of Finance.

Fundamental Theorem of Finance. *The set Π is compact and contains a strictly positive linear functional π on X.*

Proof. Notice that the convex set $K = \{x \in X_+ : x \cdot u = 1\}$ does not interest the linear subspace $Z = \{y \in Y : \varphi(y) = 0\}$. By the Strong Separation Theorem (see for instance Aliprantis and Border (1999), Theorem 5.58), there exists a non-null π in X such that, for every k in K and for every z in Z,

$$\pi \cdot k > \pi \cdot z.$$

As Z is a linear space, $\pi \cdot z = 0$ for every z in Z. If $\pi \cdot x \leq 0$ for some non-null x in X_+, then

$$0 \geq \frac{1}{x \cdot u} \pi \cdot x \geq \pi \cdot \left(\frac{1}{x \cdot u} x \right) > 0,$$

a contradiction. Hence, π is a strictly positive positive linear functional on X. At no loss of generality, it can be assumed that $\pi \cdot \bar{y} = \varphi(\bar{y}) > 0$ for some $\bar{y}$ in Y. Given any y in Y, suppose that $\varphi(y) > \pi \cdot y$. Hence,

$$\varphi \left(y - \frac{\varphi(y)}{\varphi(\bar{y})} \bar{y} \right) = 0 \text{ and } \pi \cdot \left(y - \frac{\varphi(y)}{\varphi(\bar{y})} \bar{y} \right) < 0,$$

a contradiction. This set is compact as it is contained in $\{\pi \in X_+ : \pi \cdot u = \varphi(u)\}$. □

When markets are incomplete, Π contains multiple value kernels. Nevertheless, values are restricted by upper and lower bounds.

Fundamental Theorem of Duality. *For every x in X,*

$$\max_{\pi \in \Pi} \pi \cdot x = \min_{y \in Y} \{\varphi(y) : x \leq y\}$$

and

$$\min_{\pi \in \Pi} \pi \cdot x = \max_{y \in Y} \{\varphi(y) : y \leq x\}.$$

Proof. We prove the first statement only, as the argument is specular for the other statement. Observe that, for some sufficiently large $\lambda > 0$,

$$-\lambda u \leq x \leq \lambda u,$$

where u is the strictly positive claim in Y. Thus, by no arbitrage,

$$-\lambda \varphi(u) \leq \inf_{y \in Y} \{\varphi(y) : x \leq y\} \leq \lambda \varphi(u).$$

This shows that the infimum is finite. For every n in $\mathbb{N}$, there exists a claim y^n in $\{y \in Y : x \leq y\}$ such that

$$\varphi(y^n) \leq \inf_{y \in Y} \{\varphi(y) : x \leq y\} + \frac{1}{n}.$$

If the sequence $(y^n)_{n\in\mathbb{N}}$ is bounded, then the claim follows. Otherwise, observe that $\hat{y}^n = y^n/\|y^n\|$ is also a tradable claim in Y satisfying

$$\frac{x}{\|y^n\|} \leq \hat{y}^n$$

and

$$\varphi\left(\hat{y}^n\right) \leq \frac{\inf_{y\in Y}\left\{\varphi\left(y\right) : x \leq y\right\}}{\|y^n\|} + \frac{1}{n\|y^n\|}.$$

Taking a subsequence of $(\hat{y}^n)_{n\in\mathbb{N}}$ in Y converging to $\hat{y}$ in Y, we obtain that $\hat{y} > 0$ and $\varphi(\hat{y}) \leq 0$, contradicting no arbitrage. Thus, there exists $\bar{y}$ in Y such that

$$\varphi\left(\bar{y}\right) = \min_{y\in Y}\left\{\varphi\left(y\right) : x \leq y\right\}.$$

Clearly, $\pi \cdot (x - \bar{y}) \leq 0$ for every π in Π. To prove that the opposite inequality is satisfied by some π in Π, consider the convex set C in $\mathbb{R} \times X$ defined by

$$\left\{\left(\varphi\left(\bar{y} - y\right), y - x\right) \in \mathbb{R} \times X : y \in Y\right\}.$$

This set does not intersect $\mathbb{R}_+ \times X_{++}$. Hence, by the Separating Hyperplane Theorem, there exists a non-null (μ, π) in $\mathbb{R}_+ \times X_+$ such that, for every y in Y,

$$\mu\varphi\left(\bar{y} - y\right) \leq \pi \cdot \left(x - y\right).$$

It can be verified that $\mu > 0$ and, hence, $\mu = 1$ at no loss of generality. Also,

$$0 \leq \varphi\left(\bar{y} - \bar{y}\right) \leq \pi \cdot \left(x - \bar{y}\right) \leq 0,$$

thus proving that $\pi \cdot (x - \bar{y}) = 0$. Finally, notice that, when y lies in Y, also $(\bar{y} - y)$ is in Y. It follows that

$$\varphi\left(y\right) \leq \varphi\left(\bar{y} - \left(\bar{y} - y\right)\right) \leq \pi \cdot \left(x - \left(\bar{y} - y\right)\right) \leq \pi \cdot y.$$

As Y is a linear space, it is also true that $\varphi(-y) \leq \pi \cdot (-y)$. We conclude that, for every y in Y,

$$\varphi\left(y\right) = \pi \cdot y,$$

which reveals that π is an element of Π. □

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