Pareto indivisible allocations, revealed preference and duality

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Introduction

In 1967, Afriat solved the “revealed preference” problem posed by Samuelson. Given the observation of n consumptions baskets and corresponding prices, can one rationalize these consumptions as the consumption of a single representative consumer facing different prices?

In 1974, Shapley and Scarf investigated the “housing problem”. Given an initial allocation of n houses to n individuals, and assuming individuals form preferences over houses and can trade houses, what is the core of the corresponding game? It is assumed that houses form no preferences over houses, or at least that they can’t voice them. In this setting, Shapley and Scarf showed the non-emptiness of the core, as well as an algorithm to arrive to a core allocation: David Gale’s method of “top-trading cycles”.

In this paper we shall:

- argue that both problems are *dual* in a precise sense

- give a new characterization of both problems in terms of an optimal assignment problem; and welfare theorems for Pareto efficient outcomes in the housing problem

- introduce a natural index of rationalizability

- investigate a weaker notion of rationalizability

THIS IS STILL WORK IN PROGRESS – COMMENTS AND FEEDBACKS ARE MOST WELCOME.

Related literature

Theory of revealed preference in consumer demand: problem formulated by Samuelson (1938), solved by Afriat (1967). Diewert (1973) provided a Linear Programming proof and Varian (1982) an algorithmic solution. Fostel, Scarf and Todd (2004) provided alternative proofs. Matzkin (1991) and Forges and Minelli (2009) extended the theory to nonlinear budget constraints. Geanakoplos (2006) gives a proof of Afriat's theorem using a minmax theorem.

Efficiency in the indivisible allocation problem: Shapley and Scarf (1974) formulate the "housing problem" and give an abstract characterization of the core, Roth et al (2004) study a related "kidney problem" and investigate mechanism design aspect.

Revealed preferences for matching problems: Galichon and Salanié (2010) and Echenique, SangMok and Shum (2010) investigate the problem of revealed preferences in a matching game with transferable utility.

Talk's outline

1. Pareto efficient allocations

2. Strong and weak efficiency

3. Geometric interpretation of revealed preference

1 Pareto efficient allocations

1.1 Preamble: Generalized Revealed Preference

Assume as in Forges and Minelli (2009) that consumer has budget constraint $g_i(x) \leq 0$ in experiment i, and that x_i is chosen. Assume $g_i(x_i) = 0$, generalizing Afriat (1969), in which $g_i\left(x_j\right) = x_j \cdot p_i - x_i \cdot p_i$. One would like to know whether there is a utility level v_j associated to good j such that consumption x_i results from the maximization of consumer i's utility under budget constraint $g(x) \leq 0$, namely

$$i \in \text{argmax}_j \left\{ v_j : g_i\left(x_j\right) \leq 0 \right\}.$$

By Forges and Minelli (2009), building on Fostel, Shapley and Todd (2004), the following equivalence holds:

Theorem 0. *Set* $R_{ij} = g_i\left(x_j\right)$*. Then the following conditions are equivalent:*

(i) The matrix R_{ij} *satisfies* **"cyclical consistency"***: for any cycle* $i_1, ..., i_{p+1} = i_1$*,*

$$\forall k,\ R_{i_k i_{k+1}} \leq 0 \textit{ implies } \forall k,\ R_{i_k i_{k+1}} = 0,$$

(ii) There exist numbers (v_i, λ_i)*,* $\lambda_i > 0$*, such that*

$$v_j - v_i \leq \lambda_i R_{ij},$$

(iii) There exist numbers v_i *such that*

$$R_{ij} < 0 \textit{ implies } v_j - v_i < 0.$$

Then v_j can be seen as utility level associated to good j that rationalizes the data in the sense that

$$i \in \operatorname{argmax}_j \left\{v_j : g_i\left(x_j\right) \leq 0\right\}.$$

1.2 Pareto efficient allocation of indivisible goods

Consider n indivisible goods (eg. houses) $j = 1, ..., n$ to be allocated to n individuals. Cost of allocating (eg. transportation cost) house j to individual i is c_{ij}. Assume good i is allocated to individual i. Question: when is this allocation efficient?

If there are two individuals, say i_1 and i_2 that would both benefit from swapping houses, then allocation is not efficient. Thus if allocation is efficient, then inequalities $c_{i_1 i_2} \leq c_{i_1 i_1}$ and $c_{i_2 i_1} \leq c_{i_2 i_2}$ cannot hold simultaneously unless they are both equalities. More generally, cannot have exchange rings whose members would benefit from trading (strictly for some).

We shall argue that this problem is dual to the problem of Generalized Revealed Preferences.

1.3 A dual interpretation of revealed preference

From the previous discussion, allocation is efficient if and only if for every "trading cycle" $i_1, ..., i_{p+1} = i_1$,

$$\forall k,\ c_{i_k i_{k+1}} \leq c_{i_k i_k} \text{ implies } \forall k,\ c_{i_k i_{k+1}} = c_{i_k i_k}$$

that is, introducing $R_{ij} = c_{ij} - c_{ii}$,

$$\forall k,\ R_{i_k i_{k+1}} \leq 0 \text{ implies } \forall k,\ R_{i_k i_{k+1}} = 0,$$

which is to say that allocation is efficient if and only if the matrix R_{ij} is cyclically consistent.

By the equivalence between (i) and (ii) in Theorem 0 above, allocation is efficient if and only if

$$\exists v_i \text{ and } \lambda_i > 0,\ v_j - v_i \leq \lambda_i R_{ij}. \qquad \text{(PARETO)}$$

Equilibrium in the indivisible allocation game.

Allocate house i to individual i, and let people trade. Let π_j be the price of house j. We have a No-trade equilibrium supported by prices π if any house within i's budget set is not strictly preferred to i's house. That is, we have

$$\exists \pi_i,\ \pi_j \leq \pi_i \text{ implies } R_{ij} \geq 0, \qquad \text{(EQUILIBRIUM)}$$

that is equivalently

$$R_{ij} < 0 \text{ implies } \pi_j > \pi_i$$

which is exactly formulation (iii) of Theorem 0 with $\pi_i = -v_i$.

By Theorem 0 and this interpretation, one has then

$$\text{(EQUILIBRIUM)} \iff \text{(PARETO)},$$

which under this interpretation gives us a welfare result:

Proposition 1. *In the allocation problem of indivisible goods, Pareto allocations are no-trade equilibria supported by prices, and conversely, no-trade equilibria are Pareto efficient.*

This is a "dual" interpretation of revealed preference, where v_i (utilities in generalized RP theory) become budgets here, and c_{ij} (budgets in generalized RP theory) become utilities here. To summarize this duality:

	Revealed prefs.	Pareto indiv. allocs.
setting	consumer demand	allocation problem
budget sets	$\left\{j : c_{ij} \leq c_{ii}\right\}$	$\{-v : -v \leq -v_i\}$
cardinal utilities to j	v_j	$-c_{ij}$
# of consumers	one, representative	n, $i \in \{1, ..., n\}$
# of experiments	n	one
goods	divisible	indivisible
unit of c_{ij}	dollars	utils
unit of v_i	utils	dollars
interpretation	Afriat's theorem	Welfare theorem

2 A Negishi theorem for Pareto assignments

Reminder on the optimal assignment problem. Recall the optimal assignment problem:

$$\min_{\sigma \in S} \sum_{i=1}^{n} c_{i\sigma(i)}.$$

where S is the set of permutations of $\{1, ..., n\}$. Interpretation: σ_0 minimizes utilitarian sum of cardinal welfare losses.

By Linear Programming duality (Dantzig 1939; Shapley-Shubik 1971), we get that

$$\min_{\sigma \in S} \sum_{i=1}^{n} c_{i\sigma(i)} = \max_{u_i + v_j \leq c_{ij}} \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j.$$

For σ_0 solution, there is a pair (u, v) solution to the dual problem such that

$$\begin{array}{rcl} u_i + v_j & \leq & c_{ij} \\ \text{if } j = \sigma_0(i), & & \text{then } u_i + v_j = c_{ij}. \end{array}$$

A Negishi characterization. Going back to the Pareto assignment problem, we have the following result:

Theorem 2. *In the housing problem, the following conditions are equivalent:*

(i) Allocation $\sigma_0 = Id$ *is Pareto efficient,*

(ii) Allocation $\sigma_0 = Id$ *is a No-trade equilibrium,*

(iii) $\exists \lambda_i > 0$ *and* $v \in \mathbb{R}^n$ *such that*

$$v_j - v_i \leq \lambda_i R_{ij},$$

(iv) $\exists \lambda_i > 0$ *such that*

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0,$$

that is

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i c_{i\sigma(i)} = \sum_{i=1}^{n} \lambda_i c_{ii}.$$

Remark 1. The economic interpretation for this result is quite clear. (iv) is

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i c_{i\sigma(i)} = \sum_{i=1}^{n} \lambda_i c_{ii}.$$

which means that Pareto efficient allocations coincide with the maximizers of weighted utilitarian welfare functions with positive social weights. The λ_i's can therefore be interpreted as "Negishi weights", see [Negishi (1960)].

Remark 2. The translation of the previous result in terms of revealed preference is the following:

Theorem 2'. *In the revealed preference problem, the data are rationalizable if and only if* $\exists \lambda_i > 0$ *such that*

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0$$

where $R_{ij} = g_i\left(x_j\right)$*.*

Proof of Theorem 2. As seen above the essence of equivalence between (i), (ii) and (iii) has been proven in the revealed preference literature. The new result is the equivalence between (iii) and (iv), which we now prove. One has

(iv)$\iff \exists \lambda_i > 0, \ \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0$

$\iff \exists \lambda_i > 0, \ \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}$ is reached for $\sigma = Id$

$\iff \exists \lambda_i > 0, \, u, v \in \mathbb{R}^n$

$$\begin{aligned} u_i + v_j &\leq \lambda_i R_{ij} \\ u_i + v_i &= 0 \end{aligned}$$

$\iff \exists \lambda_i > 0, \, v \in \mathbb{R}^n$

$$v_j - v_i \leq \lambda_i R_{ij},$$

which is (iii). □

3 Strong and weak rationalizability

3.1 Indices of rationalizability

It is tempting to set

$$A = \max_{\lambda \in \Delta} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}$$

where $\Delta = \left\{ \lambda \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}$.

Indeed, we have $A \leq 0$, and by compacity of Δ, equality holds if and only if there exist $\lambda \in \Delta$ such that

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0.$$

Of course, this does not work as the λ_i's in Theorem 2 need to be all positive, not simply nonnegative. For example, in the housing problem, if individual $i = 1$ has his most preferred option, then $\lambda_1 = 0$ and all the other λ_i's are zero, and $A = 0$. However, allocation may not be Pareto because there may be inefficiencies among the rest of the individuals.

Hence imposing $\lambda > 0$ is crucial. Fortunately, it turns out that one can restrict $\mathbf{\Delta}$ to a subset which is convex, compact and away from zero:

Lemma 3. *There is $\varepsilon > 0$ (dependent only on matrix R) such that the λ_i's in Theorem 2 above (if they exist) can be chosen such that*

$$\begin{cases} \lambda_i \geq \varepsilon \text{ for all } i, \\ \sum_{i=1}^{n} \lambda_i = 1. \end{cases}$$

Proof. Standard construction (see [Fostel et al. (2004)]) of the λ_i's and the v_i's provides a deterministic procedure that returns strictly positive $\lambda_i \geq 1$ within a finite and bounded number of steps, with only the entries of R_{ij} as input; hence λ, if it exists, is bounded, so there exists M depending only on R such that $\sum_{i=1}^{n} \lambda_i \leq M$. Normalizing λ so that $\sum_{i=1}^{n} \lambda_i = 1$, one sees that one can choose $\varepsilon = 1/M$. □

We denote $\mathbf{\Delta}_\varepsilon$ the set of such vectors λ, and $\mathbf{\Delta}$ the simplex $\left\{\lambda : \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^n \lambda_i = 1\right\}$. Recall $R_{ij} = c_{ij} - c_{ii}$, and introduce

$$A^* = \max_{\lambda \in \mathbf{\Delta}_\varepsilon} \min_{\sigma \in S} \sum_{i=1}^n \lambda_i R_{i\sigma(i)},$$

so that we have the following result:

Proposition 4. *We have:*

(i) $A^* = 0$ *if and only if there exist scalars* v_i *and weights* $\lambda_i > 0$ *such that*

$$v_j - v_i \leq \lambda_i R_{ij}.$$

(ii) $A = 0$ *if and only if there exist scalars* v_i *and weights* $\lambda_i \geq 0$*, not all zero, such that*

$$v_j - v_i \leq \lambda_i R_{ij}.$$

(iii) $A^* \leq A \leq 0$.

Proof. (i) follows from Lemma 3. To see (ii), note that $A = 0$ is equivalent to the existence of $\lambda \in \mathbf{\Delta}$ such that $\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0$. The rest of the proof works as the equivalence between (iii) and (iv) in Theorem 2. The inequality (iii) is immediate. □

3.2 What happens when some λ's can be zero?

Coherent subcoalition. In the housing problem, a non-empty subcoalition $I \subset \{1, ..., n\}$ is said to be *coherent* when for each of its members i, it also contains the owners of the goods with whom i would be willing to exchange. Namely, I is coherent when

$$i \in I \text{ and } R_{ij} < 0 \text{ implies } j \in I.$$

In particular, $\{1, ..., n\}$ is coherent; any coalition where individuals are assigned their top choice is also coherent.

Theorem 5. *In the housing problem, we have:*

(i) $A^* = 0$ *iff allocation* $\sigma_0 = Id$ *is Pareto efficient for the population* $\{1, ..., n\}$*,*

(ii) $A = 0$ *iff allocation* $\sigma_0 = Id$ *is Pareto efficient for some coherent subcoalition,*

and (i) implies (ii).

Before we give the proof of this result, we state its equivalent translation in terms of revealed preference.

Say that a subset of the data included in $\{1, ..., n\}$ is *coherent* if $i \in I$ and i directly revealed preferred to j implies $j \in I$.

Theorem 5'. *In the revealed preference problem, we have:*

(i) $A^* = 0$ *iff the data are rationalizable,*

(ii) $A = 0$ *iff a coherent subset of the data is rationalizable,*

and (i) implies (ii).

Proof. (i) was proved in Theorem 2 above.

Let us show the equivalence in (ii). The proof of that same theorem implies that $A = 0$ is equivalent to the existence of $\exists \lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ and $v \in \mathbb{R}^n$ such that

$$v_j - v_i \leq \lambda_i R_{ij},$$

so defining I as the set of $i \in \{1, ..., n\}$ such that $\lambda_i > 0$, this implies that allocation $\sigma_0 = Id$ is Pareto efficient for subcoalition I. We now show that I is coherent. Indeed, for any two k and l not in I and i in I, one has $v_k = v_{l'} \geq v_i$; thus if $i \in I$ and $R_{ij} < 0$, then $v_j < v_i$, hence $j \in I$, which show that I is coherent.

Conversely, assume allocation $\sigma_0 = Id$ is Pareto efficient for a coherent subcoalition I. Then there exist $(u_i)_{i \in I}$ and $(\mu_i)_{i \in I}$ such that $\mu_i > 0$ and

$$u_j - u_i \leq \mu_i R_{ij}$$

for $i, j \in I$. Complete by arbitrary values of u_i for $i \notin I$, and introduce $\tilde{R}_{ij} = 1_{\{i \in I\}} R_{ij}$. One has $\tilde{R}_{ij} < 0$

implies $i \in I$ and $R_{ij} < 0$ hence $j \in I$, thus $u_j - u_i < 0$. Hence by theorem 0, there exist v_i and $\bar{\lambda}_i > 0$ such that

$$v_j - v_i \leq \bar{\lambda}_i \tilde{R}_{ij}$$

and defining $\lambda_i = \bar{\lambda}_i \mathbf{1}_{\{i \in I\}}$, one has

$$v_j - v_i \leq \lambda_i R_{ij}$$

which is equivalent to $A = 0$.

The implication $(i) \Rightarrow (ii)$ results from inequality $A^* \leq A \leq 0$. $\square$

4 Concluding remarks

Recall λ_i is interpreted in Afriat's theory as the Lagrange multiplier of the budget constraint. Allowing for $\lambda = 0$ corresponds to excluding wealthiest individuals as outliers. Theory with $\lambda \geq 0$ is weaker, less reject. How much so empirically?

Link with Shapley-Scarf "top trading cycles" procedure: what does it imply in terms of social weights?

Similar dual interpretation for revealed preferences in matching problems (Galichon and Salanié (2010), Echenique et al. (2010))?

Continuous generalization using the theory of Optimal Transportation.

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